Lecture Notes in General Topology

Lectures by Dr Sheng-Chi Liu

Throughout these notes, □ signifies end proof, ▲ signifies end of example, and ■ marks the end of exercise.

Much of the theory herein is made easier to understand and more sensible by judicious use of pictures and graphs. The author of these notes is lazy and has not included such pictures. The reader is encouraged to try to draw them for themselves.

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Lecture 1 Laying the groundwork

There are two main purposes of topology:

First: Classify geometric objects. For instance, are $[0,1]$, $(0,1)$, and the real line the same? Are they different?

Or in higher dimension, compare the closed square $[0,1] \times [0,1]$, the open square $(0,1) \times (0,1)$ and $\mathbb{R}^2$.

Or in three dimensions, compare a sphere and a cube; a doughnut (torus) and a double doughnut (double torus).

Of course to answer this we must first determine what we *mean* when we say two objects are the same or different.

Second: To do analysis. Some recap: we talk about functions $f: \mathbb{R} \to \mathbb{R}$ or $f: I \to \mathbb{R}$, $I \subset \mathbb{R}$, being ***continuous***. Likewise $f: \mathbb{R}^n \to \mathbb{R}^n$, and so on.

In many fields we have occasion to talk about functions on more abstract spaces. Say, $f: X \to Y$: what does it mean for f to be ***continuous*** in this setting?

For example, we might walk about $X = \mathbb{Q}$, or the Galois group of $\overline{Q}/\mathbb{Q}$, or an algebraic variety, and so on.

Experience with analysis says that, in order to talk about abstract continuity in a sensible way, we need to consider a certain collection of subsets of X.

1.1 Topological space

Definition 1.1.1 (Topology)**.** Let X be a set. A collection $\mathscr{T}$ of subsets of X is a ***topology*** for X if $\mathscr{T}$ has the following properties:

(*i*) $\varnothing$ and X are in $\mathscr{T}$,

(*ii*) any union of sets in $\mathscr{T}$ is in $\mathscr{T}$, and

(*iii*) any finite intersection of sets in $\mathscr{T}$ is in $\mathscr{T}$.

Definition 1.1.2 (Topological space, open set, (open) neighbourhood)**.** A ***topological space*** is a pair $(X, \mathscr{T})$ where X is a set and $\mathscr{T}$ is a topology for X.

The sets in $\mathscr{T}$ are called ***open sets***.

An open set $U \in \mathscr{T}$ is called an ***(open) neighbourhood*** of $x \in X$ if $x \in U$.

Some trivial examples.

Example 1.1.3 (Trivial topology)**.** If $\mathscr{T} = \{ \varnothing, X \}$, then $\mathscr{T}$ is a topology on X called the ***trivial topology*** or ***indiscrete topology***.

This is useless since there is no structure at all—there are no open sets distinguishing any points. ▲

Example 1.1.4. The set $\mathscr{T}$ composed of *all* subsets of X is called the ***discrete topology*** on X.

This is useless for a different reason: *everything* is an open set, so every set in it looks geometrically the same. ▲

Our first nontrivial example:

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Example 1.1.5. Let $X = \{ a, b, c \}$. There are many different topologies for X. We have as before $\mathscr{T}_1 = \{ \varnothing, X \}$, and

$$\mathscr{T}_2 = \{ \varnothing, X, \{ a \}, \{ b \}, \{ b \}, \{ a, b \}, \{ a, c \}, \{ b, c \} \},$$

but there are more.

Clearly(?) $(X, \mathscr{T}_1) \neq (X, \mathscr{T}_2)$, even though the underlying space X is the same in both.

Next, let's consider $\mathscr{T}_3 = \{ \varnothing, X, \{ a \}, \{ a, b \} \}$. Clearly the first axiom of a topology is satisfied—trying the last two axioms we find that they, too, are satisfied.

This is not immediately obvious, because:

Remark 1.1.6. Not every collection of subsets of X is a topology for X. For instance, $\mathscr{T}_4 = \{ \varnothing, X, \{ a \}, \{ b \} \}$ is *not* a topology, since $\{ a \} \cup \{ b \} = \{ a, b \} \notin \mathscr{T}_4$, so the second axiom is violated.

Similarly, $\mathscr{T}_5 = \{ \varnothing, X, \{ a, b \}, \{ b, c \} \}$ is not a topology either, since $\{ a, b \} \cap \{ b, c \} = \{ b \} \notin \mathscr{T}_5$. ▲

Definition 1.1.7 (Closed set)**.** A subset S of X is defined to be ***closed*** if $X \setminus S$ is open (which of course implies there's an underlying topology we're working with).

By De Morgan's law, we therefore have the following properties of closed sets:

(*i*) $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ are closed,

(*ii*) any intersection of closed sets is closed, and

(*iii*) any finite union of closed sets is closed.

The following exercises review some basic set theory which will soon come in handy.

Exercise 1.1. State De Morgan's laws and verify them. ■

Exercise 1.2. (*a*) What is an equivalence relation on a set?

(*b*) Let "$\sim$" be an equivalence relation on a set A. Let C_1 and C_2 be two equivalence classes. Show that either $C_1 = C_2$ or $C_1 \cap C_2 = \varnothing$. ■

Exercise 1.3. Let X be a set and let $\mathscr{T}$ be a family of subsets U of X such that $X \setminus U$ is finite, together with the empty set $\varnothing$. Show that $\mathscr{T}$ is a topology. ($\mathscr{T}$ is called the ***cofinite topology*** of X.) ■

Exercise 1.4. (*a*) Show that $\mathbb{Q}$ is countable.

(*b*) Is the polynomial ring $\mathbb{Q}[x]$ countable? Explain why? ■

Exercise 1.5. (*a*) Let $A = \{ 0, 1 \}$. Define $A^{\mathbb{N}} := \prod_{i=1}^{\infty} A_i$, where $A_i = A$ for $i = 1, 2, \ldots$. Show that $A^{\mathbb{N}}$ is uncountable.

(*b*) Show that $\mathbb{R}$ is uncountable. ■

Exercise 1.6. Let X be a topological space. Show that if U is open in X and A is closed in X, then $U \setminus A$ is open and $A \setminus U$ is closed in X. ■

Definition 1.1.8 ((Strictly) finer topology)**.** Suppose $\mathscr{T}$ and $\mathscr{U}$ are two topologies for X. If $\mathscr{T} \supset \mathscr{U}$ (respectively $\mathscr{T} \supsetneq \mathscr{U}$), then we say that $\mathscr{T}$ is ***finer*** (respectively ***strictly finer***) than $\mathscr{U}$.

In other words, an open set in $(X, \mathscr{U})$ is also open in $(X, \mathscr{T})$, but not necessarily the other way around.

Example 1.1.9. Consider in $\mathbb{R}$ the sets $I_1 = (0,1)$, $I_2 = (\frac{1}{2}, 2)$, and $I_3 = (2,3)$ are open. But they are not the only open sets: we are allowed to take arbitrary unions and finite intersections. So, for instance, $I_1 \cup I_3 = (0,1) \cup (2,3)$ is also open. ▲

However the open sets themselves are still the *building blocks* of open sets; the open sets themselves can be complicated, like the last example, but we can *generate* all of them by only looking at open intervals.

Lecture 2 Basis

2.1 Basis for a topology

In $\mathbb{R}^2$, the basic building block of an open set is an open ball, $B(a,r) = \left\{ x \,\middle|\, |x - 1| < r \right\}$. But this is not the only open set. The key, however, is that for any point in an arbitrary open set, we can find an open ball around the point that is contained in the set.

Definition 2.1.1 (Basis)**.** Let X be a topological space. A collection $\mathscr{B}$ of open subsets of X is a ***basis*** for the topology of X if every open subset of X is a union of sets in $\mathscr{B}$.

In other words, the basis generalises the set of open balls in $\mathbb{R}^2$:

Theorem 2.1.2. *A collection $\mathscr{B}$ of open subsets of a topological space X is a basis for the topology on X if and only if for each $x \in X$ and each neighbourhood U of x, there exists $V \in \mathscr{B}$ such that $x \in V$ and $V \subset U$.*

Proof. For the forward direction, suppose $\mathscr{B}$ is a basis. Then for any open neighbourhood U of x,

$$U = \bigcup_{\alpha} V_\alpha,$$

where $V_\alpha \in \mathscr{B}$ by the definition of basis. Hence, since $x \in U$, we must have $x \in V_{\alpha_0} \in \mathscr{B}$ for some α_0, and $V_{\alpha_0} \subset U$.

For the converse, let U be an open sets of X. For each $x \in U$, by our assumption there exists some $V_x \in \mathscr{B}$ such that $x \in V_x \subset U$. Hence

$$U = \bigcup_{x \in U} V_x. \qquad \square$$

Theorem 2.1.3. *A collection $\mathscr{B}$ of subsets of X is a basis for the topology on X if and only if $\mathscr{B}$ has the following properties:*

(i) Each $x \in X$ lies in at least one set in $\mathscr{B}$.

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(*ii*) *If $U, V \in \mathscr{B}$, and $x \in U \cap V$, then there exists $W \in \mathscr{B}$ such that $x \in W$ and $W \subset U \cap V$.*

Proof. In the forward direction, (i) follows from X being open. For (ii), since $U \cap V$ is open, we must have some $W \in \mathscr{B}$ between x and $U \cap V$ by the previous theorem.

For the opposite direction, Let $\mathscr{T}$ be the collection of all subsets of X that are unions of sets in $\mathscr{B}$.

By convention, we also take $\varnothing \in \mathscr{T}$, as the empty union.

If $\mathscr{T}$ is a topology, then by definition $\mathscr{B}$ is a basis for it. Hence we claim that $\mathscr{T}$ is a topology for X.

First, if $x \in X$, by (i), $x \in U_x \in \mathscr{B}$. Hence

$$X = \bigcup_{x \in X} U_x \in \mathscr{T},$$

so the first two conditions for a topology are satisfied.

Arbitrary union of elements in $\mathscr{T}$ is in $\mathscr{T}$ by definition of $\mathscr{T}$ in this case.

We need to show that a finite intersection of sets in $\mathscr{T}$ is also in $\mathscr{T}$. Since the intersection is finite, it suffices to show that the intersection of two sets in $\mathscr{T}$ is in $\mathscr{T}$.

Suppose $U, V \in \mathscr{T}$. Since U and V are unions of sets in $\mathscr{B}$, there exist $U_0, V_0 \in \mathscr{B}$ such that $x \in U_0 \subset U$ and $x \in V_0 \subset V$. Hence $x \in U_0 \cap V_0$. Hence by (ii), there exists some $W_x \in \mathscr{B}$ such that $x \in W_x \subset U_0 \cap V_0 \subset U \cap V$.

So

$$U \cap V = \bigcup_{x \in U \cap V} W_x,$$

so by definition $U \cap V \in \mathscr{T}$, being the union of sets in $\mathscr{B}$. □

Exercise 2.1. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$ respectively, on X. Then the following are equivalent:

(*i*) $\mathscr{T}'$ is finer than $\mathscr{T}$ (i.e., $\mathscr{T} \subset \mathscr{T}'$),

(*ii*) For each $x \in X$ and $U \in \mathscr{B}$ containing x, there exists $U' \in \mathscr{B}'$ such that $x \in U' \subset U$. ■

Example 2.1.4. Let $X = \mathbb{R}$. Let $\mathscr{B}$ be the collection of all open intervals (a, b), $a < b$. The topology $\mathscr{T}$ with basis $\mathscr{B}$ is called the ***standard topology*** on $\mathbb{R}$. ▲

Example 2.1.5. Let $X = \mathbb{R}$ and $\mathscr{B}'$ be the collection of all half-open intervals $[a, b)$. The topology $\mathscr{T}'$ with basis $\mathscr{B}'$ is called the ***lower limit topology*** on $\mathbb{R}$. ▲

Note that $\mathscr{T}'$ is strictly finer than $\mathscr{T}$. I.e., $\mathscr{T}' \supsetneq \mathscr{T}$.

To see this, notice how for each $x \in \mathbb{R}$ and $x \in (a, b)$, we need to find $[c, d)$ such that $x \in [c, d) \subset (a, b)$. This is clearly easy: pick and $a < c \leq x$ and $x < d \leq b$. That it is strictly finer is clear: you can't always do the opposite: if $x = c$, there exists no (a, b) such that $c \in (a, b) \subset [c, d)$.

2.2 Subbasis

Exercise 2.2. Let X be a set and let $\mathscr{C}$ be a collection of subsets of X.

(a) There exists a unique smallest topology $\mathscr{T}$ on X such that $\mathscr{C} \subset \mathscr{T}$.

(b) Let $\mathscr{B}$ be the collection of subsets of X consisting of X, $\varnothing$, and all finite intersections of sets in $\mathscr{C}$. Show that $\mathscr{B}$ is a basis for $\mathscr{T}$ generated by $\mathscr{C}$. ■

Definition 2.2.1 (Generated topology, subbasis)**.** The topology $\mathscr{T}$ is called the ***topology generated*** by $\mathscr{C}$ and $\mathscr{C}$ is called a ***subbasis*** for $\mathscr{T}$.

Lecture 3 Many basic notions

3.1 Subspaces

Definition 3.1.1 (Subspace, subspace topology)**.** Let $(X, \mathscr{T})$ be a topological space. Let Y be a subset of X. Then the collection

$$\mathscr{T}_Y = \left\{ Y \cap U \,\middle|\, U \in \mathscr{T} \right\}$$

is a topology on Y called the ***subspace topology*** or ***relative topology***.

We call $(Y, \mathscr{T}_Y)$ a ***subspace*** of $(X, \mathscr{T})$.

Remark 3.1.2. A (relative) open subset of Y need not be open in X.

Example 3.1.3. Take $X = \mathbb{R}$, and $Y = [0, \infty)$. The set $(-1, 1)$ is open in X, but the relative open set in Y is $[0, 1)$, which is not open in X. ▲

Lemma 3.1.4. *Let Y be a subspace of X. If U is (relatively) open in Y and Y is open in X, then U is open in X.*

Proof. By definition, U being open in Y means $U = Y \cap V$ for some V open in X. Moreover Y is open in X, so by definition of topology, $Y \cap V = U$ is open in X, being the finite intersection of open sets in X. □

Exercise 3.1. Let Y be a subspace of X. If $\mathscr{B}$ is a basis for the topology on X, then

$$\mathscr{B}_Y = \left\{ U \cap Y \,\middle|\, U \in \mathscr{B} \right\}$$

is a basis for the subspace topology on Y. ■

Theorem 3.1.5. *Let Y be a subspace of X. Then a subset E of Y is (relatively) closed in Y if and only if E is the intersection of Y and a closed subset of X.*

Proof. Assume E is relatively closed in Y. In other words, $Y \setminus E$ is open in Y. Hence $Y \setminus E = Y \cap U$ for some open subset U of X.

Take complements (in X) and we get

$$E = Y \cap (X \setminus U)$$

Since U is open in X, $X \setminus U$ is open.

For the converse direction, assume $E = Y \cap C$ where C is a closed subset of X. Hence $Y \setminus E = Y \cap (X \setminus C)$, where $X \setminus C$ is open in X. Hence $Y \setminus E$ is relatively open in Y, making E closed in Y. □

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3.2 Interior and closure

Definition 3.2.1 (Interior, closure)**.** Let X be a topological space. Let $S \subset X$ be a subset.

(i) The ***interior*** $\mathring{S}$ or $\operatorname{int}(S)$ is defined as the union of all open sets contained in S.

(ii) The ***closure*** $\overline{S}$ of S is defined as the intersection of all closed sets containing S.

Remark 3.2.2. Since arbitrary unions of open sets is open, $\operatorname{int}(S)$ is open. Similarly, arbitrary intersections of closed sets is closed, so $\overline{S}$ is closed.

Exercise 3.2. Let X be a topological space and let $S \subset X$. Then $x \in \overline{S}$ if and only if every open neighbourhood U of x intersects S. ■

Theorem 3.2.3. *Let Y be a subspace of X and let E be a subset of Y. Then the relative closure of E in Y is $\overline{E} \cap Y$, where $\overline{E}$ is the closure of E in X.*

Proof. Let B denote the closure of E in Y. In other words, we want to show that $B = \overline{E} \cap Y$.

Since $\overline{E}$ is closed in X, the set $\overline{E} \cap Y$ is closed in Y, and this set contains E. Hence by definition of closure, $B \subset (\overline{E} \cap U)$.

On the other hand, B is closed in Y, meaning that $B = Y \cap C$ where C is closed in X. Also, $E \subset B$. Hence $E \subset C$. Hence $\overline{E} \subset C$. Take intersection, and we have

$$\overline{E} \cap Y \subset C \cap Y = B.$$

Hence $B = \overline{E} \cap Y$. □

We can generalise the notion of a convergent sequence to topological spaces, even if we don't have a metric:

Definition 3.2.4 (Limit point, convergence, isolated point)**.** Let X be a topological space.

(i) Let $S \subset X$ be a subset. A point $x \in X$ is called a ***limit point*** of S if every neighbourhood of x intersects with S in some point other than x.

(ii) A sequence $\{ a_n \} \subset X$ ***converges*** to $x \in X$ if for every neighbourhood U of x, there exists some $N \in \mathbb{N}$ such that $a_n \in U$ for all $n > N$.

(iii) A point $s \in S$ is called an ***isolated point*** of S if there exists a neighbourhood U of s such that $U \cap S = \{ s \}$.

Exercise 3.3. Let $\mathscr{T}$ be the cofinite topology on $\mathbb{Z}$.

(a) Show that the sequence $\{ n \}_{n=1}^{\infty}$ converges in $(\mathbb{Z}, \mathscr{T})$ to each point in $\mathbb{Z}$.

(b) Describe the convergent sequences in $(\mathbb{Z}, \mathscr{T})$. ■

Proposition 3.2.5. *Let X be a topological space and let $S \subset X$ be a subset. Then $\overline{S}$ is the union of the set of limit points of S and the isolated points of S.*

Exercise 3.4. Prove Proposition 3.2.5. ■

An immediate consequence of this is:

Corollary 3.2.6. *A subset $S \subset X$ is closed if and only if S contains all of its limit points.*

3.3 Hausdorff space

Definition 3.3.1 (Hausdorff space)**.** A topological space X is called a ***Hausdorff space*** or ***T_2-space*** if for each pair $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists neighbourhoods U_1 and U_2 of x_1 and x_2 respectively, such that $U_1 \cap U_2 = \varnothing$.

Theorem 3.3.2. *Let X be a Hausdorff space. Then every single-point set is closed.*

Proof. Let $E = \{ x_0 \}$. We want to show that $X \setminus E$ is open.

If $x \in X \setminus E$, then since X is Hausdorff there exist open sets U_x and V_x such that $x_0 \in U_x$ and $x \in V_x$ with $U_x \cap V_x = \varnothing$. Hence $V_x \subset X \setminus E$.

Taking the union of all such x,

$$X \setminus E = \bigcap_{x \in X \setminus E} V_x,$$

an arbitrary union of open sets, so $X \setminus E$ is open, whence E is closed. □

Exercise 3.5. Show that X is Hausdorff if and only if the diagonal set $\triangle = \{ (x,x) \mid x \in X \}$ is a closed unset of $X \times X$. ■

Exercise 3.6. Let A be a subset of a topological space X. Let Y be a Hausdorff space. Suppose $f \colon A \to Y$ be a continuous function that has a continuous extension to $g \colon \overline{A} \to Y$ (i.e., $g(a) = f(a)$ for all $a \in A$). Show that g is uniquely determined by f. ■

One way to think of Hausdorff spaces is that it guarantees the existence of *small* open neighbourhoods.

Theorem 3.3.3. *Let X be a Hausdorff space and let $S \subset X$ be a subset. Then $x \in X$ is a limit point of S if and only if every neighbourhood of x contains infinitely many points of S.*

Proof. The converse direction follows from the definition of limit point.

For the forward direction, suppose there exists a neighbourhood U of x such that $S \cap U$ contains only finitely many points. Thus

$$U \cap (S \setminus \{ x \}) = \{ a_1, a_2, \ldots, a_n \}$$

is some finite set. By Theorem 3.3.2, $\{ a_1, a_2, \ldots, a_n \}$, being the finite intersection of closed sets $\{ a_i \}$, is closed. Hence $X \setminus \{ a_1, a_2, \ldots, a_n \}$ is open.

Take

$$V \coloneqq U \cap (X \setminus \{ a_1, a_2, \ldots, a_n \}).$$

This is open, and $x \in V$. Hence $V \cap (S \setminus \{ x \}) = \varnothing$.

But this contradicts x being a limit point of S, since we found a neighbourhood of V that doesn't meet S in a point different from x. □

3.4 Finite product spaces

Definition 3.4.1 (Product topology)**.** Let $X_1, X_2, \ldots, X_n$ be topological spaces. The ***product topology*** on $X_1 \times X_2 \times \cdots \times X_n$ is the topology for which a basis $\mathscr{B}$ of open sets is given by

$$\mathscr{B} = \{ U_1 \times U_2 \times \cdots \times U_n \mid U_i \text{ is open in } X_i \text{ for } i = 1, 2, \ldots \}.$$

We should check that this satisfies the criterion for being a basis in Theorem 2.1.3. The first one, $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n \in \mathscr{B}$, is trivial. Secondly, if

$$(x_1, x_2, \ldots, x_n) \in (U_1 \times U_2 \times \cdots \times U_n) \cap (V_1 \times V_2 \times \cdots \times V_n),$$

then the point is also in $(U_1 \cap V_1) \times \cdots \times (U_n \cap V_n)$. Each of the constituent factors is open, and the product is open, so this is in $\mathscr{B}$. Clearly, this product of intersections is contained in the intersection of products.

Proposition 3.4.2. *Let $\mathscr{B}_i$ be a basis for the topology on X_i, $i = 1, 2, \ldots, n$. Then*

$$\mathscr{C} = \left\{ V_1 \times V_2 \times \cdots \times V_n \;\middle|\; V_i \in \mathscr{B}_i,\ i = 1, 2, \ldots, n \right\}$$

is a basis for the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

Proof. We apply Theorem 2.1.2. Let $x = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$. Let W be an open neighbourhood of x.

Then there exists a basis element $U_1 \times U_2 \times \cdots \times U_n$ such that U_i is open in X_i and $x_i \in U_i$.

Since $\mathscr{B}_i$ is a basis for X_i, there exists $V_i \in \mathscr{B}_i$ so that $x \in V_i \subset U_i$. Thus

$$x \in V_1 \times V_2 \times \cdots \times V_n \subset U_1 \times U_2 \times \cdots \times U_n \subset W$$

where $V_1 \times V_2 \times \cdots \times V_n \in \mathscr{C}$. Hence by Theorem 2.1.2, $\mathscr{C}$ is a basis for the product topology on $X_1 \times X_2 \times \cdots \times X_n$. □

Exercise 3.7. Let X and Y be topological spaces. Define the projections $\pi_1 \colon X \times Y \to X$ by $\pi_1(x, y) = x$ and $\pi_2 \colon X \times Y \to Y$ by $\pi_2(x, y) = y$. Then

$$\mathscr{B} = \left\{ \pi_1^{-1}(U) \;\middle|\; U \text{ open in } X \right\} \cup \left\{ \pi_2^{-1}(V) \;\middle|\; V \text{ open in } Y \right\}$$

is a subbasis for the topology of $X \times Y$. So the projection maps pull back open sets to open sets in the product topology.

In other words, the product topology makes the projection map continuous. In a sense, this is a better description of the product topology, because it works for infinite products too—more on that in future. ■

Lecture 4 Continuity

4.1 Continuous functions

Recall from analysis:

As a local condition, a function $f \colon \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

As a global condition, $f \colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if f is continuous at all points in $\mathbb{R}$. An important result from analysis is that this is equivalent to $f^{-1}(V)$ is open in $\mathbb{R}$ for any open $V \subset \mathbb{R}$.

This second option is the correct/productive way to generalise continuity to arbitrary topological spaces, since all we have is a notion of openness:

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Definition 4.1.1 (Continuous function)**.** Let X and Y be topological spaces.

(i) A function $f\colon X \to Y$ is ***continuous*** if $f^{-1}(V)$ is open in X for any open V in Y.

(ii) A function $f\colon X \to Y$ is ***continuous at the point*** $x_0 \in X$ if for any open subset $V \subset Y$ such that $f(x_0) \in V$, there exists an open $U \subset X$ such that $x_0 \in U$ and $f(U) \subset V$.

The correspondence in the analysis case still works:

Theorem 4.1.2. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is continuous if and only if f is continuous at every point of X.*

Proof. For the forward direction, let $x_0 \in X$ and let V be an open neighbourhood of $f(x_0)$. Since f is continuous, $f^{-1}(V)$ is open in X. Taking $U = f^{-1}(V)$, we see that $x_0 \in U$ and $f(U) \subset V$.

For the converse, let $V \subset Y$ be open. Let $x \in f^{-1}(V)$. Since f is continuous at x, there exists an open neighbourhood U_x of x such that $f(U_x) \subset V$. Let

$$U = \bigcup_{x \in f^{-1}(V)} U_x,$$

a union of open sets, hence open. In addition, $x \in f^{-1}(V)$, $x \in U_x \subset U$, so $f^{-1}(V) \subset U$. Moreover, $f(U_x) \subset V$ for each x implies $f(U) \subset V$, so $f^{-1}(V) \supset V$. Thus $f^{-1}(V) = U$, open in X. $\square$

Theorem 4.1.3. *Let X and Y be topological spaces. Let $f\colon X \to Y$. Then the following are equivalent:*

(i) *f is continuous.*

(ii) *For any subset $E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$.*

(iii) *For any closed $C \subset Y$, $f^{-1}(C)$ is closed in X.*

Proof. First, (i) implies (ii). We need to show that for $x \in \overline{E}$, $f(x) \in \overline{f(E)}$. Let V be an open neighbourhood of $f(x)$—we want to show V meets $f(E)$. Now $\in f^{-1}(V)$ is an open neighbourhood of $x \in X$. Since $x \in \overline{E}$, $f^{-1}(V) \cap E \neq \varnothing$. Hence there exists some $x_0 \in f^{-1}(V) \cap E$, and therefore $f(x_0) \in V \cap f(E)$. Thus $V \cap f(E) \neq \varnothing$, and $f(x) \in \overline{f(E)}$.

Next, (ii) implies (iii). Let $C \subset Y$ be closed. Set $E = f^{-1}(C)$. To show that E is closed, it suffices to show $\overline{E} \subset E$. For $x \in \overline{E}$, we have $f(x) \in f(\overline{E}) \subset \overline{f(E)}$ by (ii). This is contained in $\overline{C} = C$, so $f(x) \in C$ and hence $x \in f^{-1}(C) = E$.

Finally, (iii) implies (i). Let $U \subset Y$ be open. Then $C = Y \setminus U$ is closed. By assumption $f^{-1}(C)$ is closed in X. Hence

$$f^{-1}(C) = f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U),$$

so $f^{-1}(U)$ is open. $\square$

Proposition 4.1.4. *Let X_1, X_2, X_3 be topological spaces. Let $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_3$ be continuous. Then $g \circ f\colon X_1 \to X_3$ is continuous.*

Proof. This is obvious: pull an open set $U \subset X_3$ back to X_1 via X_2. $\square$

Exercise 4.1. Let X be a topological space and let $\{ E_\alpha \}_{\alpha \in \Lambda}$ be a collection of subsets such that $X = \bigcup_{\alpha \in \Lambda} E_\alpha$. Let $f\colon X \to Y$. Suppose $f\big|_{E_\alpha}$ is continuous for all $\alpha \in \Lambda$.

(a) Show that if Λ is finite and E_α is closed for all $\alpha \in \Lambda$, then f is continuous.

(b) Give an example where Λ is countable and E_α is closed for all $\alpha \in \Lambda$, but f is not continuous.

(c) Suppose $\{ E_\alpha \}_{\alpha \in \Lambda}$ is ***locally finite***, that is each $x \in X$ has a neighbourhood that intersects only finitely many E_α, and E_α is closed for all $\alpha \in \Lambda$. Show that f is continuous. ■

4.2 Homeomorphism

Definition 4.2.1 (Homeomorphism)**.** Let X and Y be topological spaces. Let $f\colon X \to Y$ be one-to-one and onto. The map f is called a ***homeomorphism*** if both f and f^{-1} are continuous.

In other words, f is a homeomorphism if is has the property that U is open in X if and only if $f(U)$ is open in Y.

Definition 4.2.2 (Homeomorphic spaces)**.** The topological spaces X and Y are ***homeomorphic*** if there exists a homeomorphism $f\colon X \to Y$.

Classifying objects and/or spaces as homeomorphic or not is an important question in topology.

For example, the cube and the sphere are homeomorphic, but the sphere and the torus are not. This is very much the realm of algebraic topology, assigning an algebraic quantity/number to a topological space that is invariant under homeomorphism.

Definition 4.2.3 (Topological property)**.** A property of a topological space is called a ***topological property*** if is is preserved under homeomorphism.

Example 4.2.4. ***Metrisability*** (being able to assign a metric) is a topological property. ▲

Example 4.2.5. Let $f\colon (-1,1) \to \mathbb{R}$ be defined by

$$f(x) = \frac{x}{1-x^2}.$$

This map is a homeomorphism, showing that $(-1,1)$ is homeomorphic to $\mathbb{R}$. ▲

4.3 Constructing continuous functions

Proposition 4.3.1. *Let X and Y be topological spaces.*

(i) *If $f\colon X \to Y$ is continuous and $Z \subset X$ is a subspace, then $f\big|_Z\colon X \to Y$ is continuous.*[1]

[1]By $f\big|_Z$ we mean f restricted to Z.

(*ii*) *If* $X = \bigcup_\alpha V_\alpha$, V_α *open and* $f\big|_{V_\alpha}$ *is continuous, then* $f\colon X \to Y$ *is continuous.*

Proof. (*i*) Let $U \subset Y$ be open. We have

$$(f\big|_Z)^{-1}(U) = f^{-1}(U) \cap Z$$

is open in Z (using the subspace topology) since $f^{1}(U)$ is open in X. Hence $f\big|_Z$ is continuous.

(*ii*) Let $U \subset Y$ be open. Then

$$X = \bigcup_\alpha V_\alpha \supset f^{-1}(U) = \bigcup_\alpha (V_\alpha \cap f^{-1}(U)) = (f\big|_{V_\alpha})^{-1}(U).$$

The right-hand side is open in X since V_α is, so $f^{-1}(U)$ is open in X. □

Theorem 4.3.2 (The pasting lemma)**.** *Let* $X = A \cup B$*, where* A *and* B *are closed in* X*. Let* $f\colon A \to Y$ *and* $g\colon B \to Y$ *be continuous. Suppose* $f(x) = g(x)$ *for all* $x \in A \cap B$*. Then*

$$h\colon X \to Y$$

defined by $h(x) = f(x)$ *if* $x \in A$ *and* $h(x) = g(x)$ *if* $x \in B$ *is a continuous map.*

Proof. Let $C \subset Y$ be closed. Since f is continuous, $f^{-1}(C)$ is closed in A. Also, A is closed in X. Hence $f^{-1}(C)$ is closed in X. Similarly for g: $g^{-1}(C)$ is closed in X.

Now

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$$

is closed in X since it is the union of two closed sets. Hence h is continuous. □

Theorem 4.3.3. *Let* $f\colon Z \to X \times Y$ *be given by* $f(z) = (f_1(z), f_2(z))$*, where* $f_1\colon Z \to X$ *and* $f_2\colon Z \to Y$*. (These maps* f_1, f_2 *are called* **coordinate functions** *of* f*.) Then* f *is continuous if and only if* f_1, f_2 *are continuous.*

Proof. Note how the projection maps

$$\begin{aligned} \pi_1\colon X \to Y \to X, &\qquad \pi_1(x, y) = x \\ \pi_2\colon X \to Y \to Y, &\qquad \pi_2(x, y) = y \end{aligned}$$

are continuous, since $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$ if U is open in X.

Now this becomes easy: Assume f is continuous. Then $f_1(z) = \pi_1(f(z))$ is continuous by Proposition 4.1.4. Similarly for f_2.

For the converse, take basis elements $U \times V$ for the topology on $X \times Y$. We need to show $f^{-1}(U \times V)$ is open. We have

$$f^{-1}(U \times V) = f_1^{-1}(Y) \cap f_2^{-1}(V).$$

Each piece is open in Z since f_1 and f_2 are continuous. Hence $f^{-1}(Y \times V)$ is open, so f is continuous. □

Lecture 5 Infinite product spaces

Let $\{ x_\alpha \}_{\alpha \in I}$ be an indexed family of topological spaces. Define

$$X := \prod_{\alpha \in I} X_\alpha = \{ x = (x_\alpha)_{\alpha \in I} \mid x_\alpha \in X_\alpha \}.$$

This index set I could be finite, infinite, countable, uncountable, etc.

We want to define a topology on X.

5.1 Box topology

This is the topology generated by the basis

$$\mathscr{B} = \left\{ \prod_{\alpha \in I} U_\alpha \;\middle|\; U_\alpha \text{ is open in } X_\alpha \right\}.$$

This is indeed a basis for a topology on X, called the ***box topology***.

This seems very natural, and follows the definition we gave for the product topology of finite products. However, there's a problem:

Remark 5.1.1. This topology contains too many open sets, and as a result it is not very useful.

By not useful we mean that many expected things break down:

Example 5.1.2. Assume X_α are compact for all $\alpha \in I$. The product space

$$\prod_{\alpha \in I} X_\alpha$$

need not be compact, under the box topology. ▲

Example 5.1.3. Assume X_α are connected for all $\alpha \in I$. The product might be ***totally disconnected***, meaning the connected component contains only one point. ▲

All by way of saying: we want a better topology.

5.2 Product topology

This time, we take the basis $\mathscr{B}$ to be all sets of the form

$$\prod_{\alpha \in I} U_\alpha,$$

with U_α open in X_α for all $\alpha \in I$, like with the box topology, only this time $U_\alpha = X_\alpha$ for all but finitely many α.

Hence the topology on X generated by $\mathscr{B}$, called the ***product topology***, has much fewer open sets than the box topology (if I is infinite), i.e.:

Remark 5.2.1. If the index set I is finite, then the box topology and the product topology coincide.

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Remark 5.2.2. Since this is much more useful than the box topology, we shall assume that

$$X = \prod_{\alpha} X_\alpha$$

is endowed with the product topology unless otherwise specified.

Define the projection map

$$\pi_\alpha \colon X \to X_\alpha$$

by $\pi_\alpha(x) = x_\alpha$, where $x = (x_\alpha)_{\alpha\in I}$.

Then

$$\mathscr{B} = \left\{ U \middle| U = \pi_{\alpha_1}^{-1}(U_\alpha) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_m}^{-1}(U_{\alpha_m}) \text{ for finitely many } \alpha_1, \ldots, \alpha_m \right\}.$$

Hence $\pi_\alpha^{-1}(U_\alpha)$ is open in X, meaning:

Proposition 5.2.3. (*i*) *The product topology on* $\prod_{\alpha\in I} X_\alpha$ *is the smallest topology such that* π_α, $\alpha \in I$, *is continuous.*

(*ii*) *If* X_α *is Hausdorff for all* $\alpha \in I$, *then* $\prod_{\alpha\in I} X_\alpha$ *is also Hausdorff in both the box topology and the product topology.*

(*iii*) *Let*

$$f \colon Z \to \prod_{\alpha\in I} X_\alpha$$

be given by $f(z) = (f_\alpha(z))_{\alpha\in I}$ *where* $f_\alpha \colon Z \to X_\alpha$. *Then* f *is continuous if and only if* f_α *is continuous for all* $\alpha \in I$.

Exercise 5.1. Prove Proposition 5.2.3. ■

5.3 Metric topology

Definition 5.3.1 (Metric)**.** A ***metric*** on a set X is a function $d \colon X \times X \to \mathbb{R}$ that has the following properties:

(*i*) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(*ii*) $d(x, y) = d(y, x)$, i.e., d is symmetric; and

(*iii*) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, i.e., d satisfies the ***triangle inequality***.

Given $\varepsilon > 0$, $x \in X$, define

$$B(x, \varepsilon) = \left\{ y \in X \;\middle|\; d(x, y) < \varepsilon \right\},$$

the ε-***ball*** centred at x.

Definition 5.3.2 (Metric topology)**.** Let d be a metric on a set X. The collection of $B(x, \varepsilon)$ for all $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the ***metric topology*** on X induced by d.

Definition 5.3.3 (Metric space)**.** Let X be a topological space.

(*i*) X is called ***metrisable*** if there exists a metric d on X that induces the topology on X.

(*ii*) A ***metric space*** X is a metrisable space X together with a a specific metric d that induces the topology.

So the metrisability of a topological space, whether a metric exists, is a topological problem. On the other hand, if you start with a metric space, understanding that space is an analysis problem.

Exercise 5.2. Assume that X is metrisable and X is homeomorphic to Y. Show that Y is also metrisable (i.e. metrisability is a ***topological property***). ■

Definition 5.3.4 (Bounded)**.** Let (X, d) be a metric space. A subset $S \subset X$ is ***bounded*** if there exists some $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in S$.

Proposition 5.3.5. *The limit of a convergent sequence in a metric space is unique.*

Proof. This is analysis: Suppose $x_n \to x$ and $x_n \to y$. We have

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0$$

by the triangle inequality, meaning that $d(x, y) = 0$, so $x = y$. □

5.4 Normed linear spaces

Many useful metric spaces are vector spaces endowed with a metric that arises from a norm, e.g., $\mathbb{R}^n$.

Definition 5.4.1 (Norm)**.** Let X be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$. A ***norm*** is a function $\|\cdot\| \colon X \to \mathbb{R}$ such that

(*i*) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(*ii*) $\|cx\| = |c|\|X\|$ for all $x \in X$ and $c \in F$; and

(*iii*) we have the ***triangle inequality***,

$$\|x + y\| \leq \|x\| + \|y\|$$

for all $x, y \in X$.

Norms induce metrics:

Lemma 5.4.2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then $d(x, y) \coloneqq \|x - y\|$ for $x, y \in X$ is a metric on X.*

Proof. Clearly $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only of $\|x - y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$.

The metric is symmetric since

$$d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = d(y, x).$$

Finally

$$\begin{aligned} d(x, z) &= \|x - z\| = \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z). \end{aligned}$$

□

Definition 5.4.3 (Normed topology)**.** Let $(X, \|\cdot\|)$ be a normed linear space. The topology induced by the metric $d(x,y) = \|x - y\|$ is called the ***normed topology*** of x induced by $\|\cdot\|$.

Example 5.4.4. The Euclidean space $\mathbb{R}^n$ is a dimension n vector space over $\mathbb{R}$. There are several common norms and metrics on this:

(*i*) The Euclidean norm and metric: For $x = (x_1, x_2, \ldots, x_n)$, we have the usual Euclidean norm

$$\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2},$$

which induces the usual Euclidean metric

$$d(x,y) = \|x - y\| = \big((x_1 - y_2)^2 + \cdots + (x_n - y_n)^2\big)^{1/2}.$$

(*ii*) The p-norm, $\|\cdot\|_p$, for $1 \le p < \infty$, defined by

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p},$$

so that for $p = 2$, $\|\cdot\|_2 = \|\cdot\|$.

(*iii*) The supremum norm, $\|\cdot\|_\infty$:

$$\|x\|_\infty = \max_i |x_i|. \qquad \blacktriangle$$

Here's a natural and interesting question: we have one and the same set $X = \mathbb{R}^n$, with three different kinds of norms and metrics. How do their open sets, in the induced metric topologies, relate?

The answer is that the topologies are the same, which we will discover below.

Definition 5.4.5 (Equivalent norms)**.** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space X. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are ***equivalent*** if there exist constants c and k such that

$$c\|x\|_1 \le \|x\|_2 \le k\|x\|_1$$

for all $x \in X$. Note that the constants c and k do *not* depend on x.

Theorem 5.4.6. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space X. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if they induce the same topology on X.*

Proof. Assume the norms are equivalent, i.e.,

$$c\|x\|_1 \le \|x\|_2 \le k\|x\|_1$$

for all $x \in X$. Then for any $\varepsilon > 0$,

$$B_1\Big(x, \frac{\varepsilon}{k}\Big) = \Big\{\, y \in X \;\Big|\; \|x - y\|_1 < \frac{\varepsilon}{k} \,\Big\} \subset \big\{\, y \in X \;\big|\; \|x - y\|_2 < \varepsilon \,\big\} = B_2(x, \varepsilon).$$

Similarly,

$$B_2(x, c\varepsilon) \subset B_1(x, \varepsilon).$$

Given $B_2(x_0, \varepsilon)$, open in $(X, \|\cdot\|_2)$, we want to show that $B_2(x_0, \varepsilon)$ is open in $(X, \|\cdot\|_1)$.

For $y \in B_2(x_0, \varepsilon)$, there exists $\varepsilon_y > 0$ such that $B_2(y, \varepsilon_y) \subset B_2(x_0, \varepsilon)$ since the ball is open in $\|\cdot\|_2$.

We know $B_1(y, \frac{\varepsilon_y}{k}) \subset B_2(y, \varepsilon_y)$, where notably the first ball is open in $(X, \|\cdot\|_1)$. Thus

$$B_2(x_0, \varepsilon) = \bigcup_{y \in B_2(x_0, \varepsilon)} B_1\Big(y, \frac{\varepsilon_y}{k}\Big)$$

is open in $(X, \|\cdot\|_1)$.

Similarly, any open set in $(X, \|\cdot\|_1)$ is also open in $(X, \|\cdot\|_2)$. Hence the two spaces have the same topology.

For the converse, assume the topologies are the same. Consider $B_1(0, 1) = \{ x \in X \mid \|x\|_1 < 1 \}$. This is open in both $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ since the topologies are equal.

In particular, there exists $r > 0$ such that

$$B_2(0, r) = \{ x \in X \mid \|x\|_2 < r \} \subset B_1(0, 1).$$

In other words, if $\|x\|_2 < r$, then $\|x\|_1 < 1$. Take $C = \frac{r}{2}$. Then if $\|x\|_2 \leq C$ implies $\|x\|_1 < 1$.

Then for any $x \neq 0$ in X, consider $y = \frac{cx}{\|x\|_2}$. Then $\|y\|_2 = c$. Hence $\|y\|_1 < 1$, so

$$\left\| \frac{cx}{\|x\|_2} \right\| < 1,$$

which rearranged means

$$c\|x\|_1 < \|x\|_2$$

for all $x \neq 0$ (the $x = 0$ case is trivial).

The same argument shows

$$\|x\|_2 \leq k\|x\|_1$$

for some $k > 0$. Hence the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. $\square$

Lecture 6 Connectedness and compactness

6.1 Norms on finite dimensional vector spaces

Theorem 6.1.1. *If X is a finite dimensional vector space, then any two norms on X are equivalent.*

Proof. Let $\{ e_1, e_2, \ldots, e_n \}$ be a basis for X over the field F. For

$$x = \sum_{i=1}^{n} c_i e_i,$$

$c_i \in F$, define the ***sup-norm***

$$\|x\|_\infty \coloneqq \max\{ |c_1|, |c_2|, \ldots, |c_n| \}.$$

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Let $\|\cdot\|$ be any norm on X. We want to show that $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. If so, then any two norms on X induce the same topology on X, which in turn implies any two norms are equivalent.

By the triangle inequality,

$$\|x\| = \left\|\sum_{i=1}^{n} c_i e_i\right\| \leq \sum_{i=1}^{n} |c_i| \|e_i\| \leq K\|x\|_\infty$$

for all $x \in X$ by taking

$$K = \sum_{i=1}^{n} \|e_i\|.$$

In other words,

$$\|x\| \leq K\|x\|_\infty$$

for all $x \in X$.

Next we need to show that there exists some $c > 0$ such that $c\|x\|_\infty \leq \|x\|$ for all $x \in X$. This requires two things from analysis which we will also discuss later:

Definition 6.1.2 (Compact set)**.** A set is ***compact*** if every open cover has a finite subcover.

A continuous function $f \colon X \to \mathbb{R}$ has a maximum and a minimum on a compact set.

With this in mind, let $B = \left\{ x \middle| \|x\|_\infty = 1 \right\}$. Then B is compact in $(X, \|\cdot\|_\infty)$.

Now we claim that $\|\cdot\| \colon (X, \|\cdot\|_\infty) \to \mathbb{R}$ is continuous under $\|\cdot\|_\infty$.

To see this, given $\varepsilon > 0$ we need to show that there exists $\delta > 0$ such that $\|x - y\|_\infty < \delta$ implies $\big|\|x\| - \|y\|\big| < \varepsilon$. By the triangle inequality,

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| < K\|x - y\|_\infty < \varepsilon$$

by taking $\delta = \frac{\varepsilon}{K}$. Since $\|\cdot\| \colon X \to \mathbb{R}$ is continuous and B is compact with respect to $\|\cdot\|_\infty$, the norm $\|\cdot\|$ must have a minimum on B, say $c > 0$.

In other words, if $\|x\|_\infty = 1$, then $\|x\| \geq c$. Then for any nonzero $x \in X$, consider $y = \frac{x}{\|x\|_\infty}$. Then $\|y\|_\infty = 1$, so $\|y\| \geq c$, whereby

$$\left\| \frac{x}{\|x\|_\infty} \right\| \geq c,$$

or in other words

$$\|x\| \geq c\|x\|_\infty. \qquad \square$$

We leave it as an exercise to check the two details left out in the above argument:

Exercise 6.1. Verify that $\|\cdot\|_\infty$ is indeed a norm on X. ■

Exercise 6.2. Verify that $B = \left\{ x \;\middle|\; \|x\|_\infty = 1 \right\}$ is compact in $(X, \|\cdot\|_\infty)$. ■

Corollary 6.1.3. *All norms on $\mathbb{R}^n$ are equivalent. I.e., $\mathbb{R}^n$ has a unique norm topology.*

6.2 Uniform convergence

Definition 6.2.1 (Uniform convergence)**.** Let X be a topological space and (Y, d) be a metric space. Let $f_n\colon X \to Y$, $n = 1, 2, 3, \ldots$. We say $\{ f_n \}$ ***converges uniformly*** to a function $f\colon X \to Y$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for all $n \geq N$ and all $x \in X$.

Theorem 6.2.2. *Let X be a topological space and (Y, d) be a metric space. Let $f_n\colon X \to Y$ be continuous, $n = 1, 2, 3, \ldots$. Suppose $\{ f_n \}$ converges uniformly to f. Then f is continuous.*

Proof. Let V be open in Y. We need to show $f^{-1}(V)$ is open in X. It suffices to show that for any $x_0 \in f^{-1}(V)$, there exists a neighbourhood U of x_0 such that $U \subset f^{-1}(V)$, which is equivalent to $f(U) \subset V$.

Let $y_0 = f(x_0) \in V$ in Y. Choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset V$, which is doable since V is open. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \frac{\varepsilon}{4}$ for all $n \geq N$ and all $x \in X$.

Since f_N is continuous at x_0, there exists a neighbourhood U of x_0 such that $f_N(U) \subset B(f_N(x_0), \frac{\varepsilon}{2}) \subset V$.

We claim that $f(U) \subset B(y_0, \varepsilon) \subset V$. This is easy: it's the triangle inequality! For $x \in U$,

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_N(x)) + f(d_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Hence $f(x) \in B(y_0, \varepsilon)$. ☐

We know from analysis that if the convergence is not uniform, the limiting function needn't be continuous, but it might be continuous on some set.

So, question: Let $\{ f_n \}$ be a sequence of continuous functions. Suppose $f_n(x) \to f(x)$ pointwise for all $x \in X$. How large is the set

$$\{ x \in X \mid f \text{ is continuous at } x \}?$$

The answer is this: if X is a compact Hausdorff space or a complete metric space, then this set is dense in X. We will come back to this problem later. (This is a consequence of the Baire category theorem.)

6.3 Connectedness

Definition 6.3.1 (Connected, disconnected)**.** (i) A topological space X is called ***disconnected*** if there exist open sets U and V such that $X = U \sqcup V$, i.e. $U, V \neq \varnothing$ and $U \cap V = \varnothing$.

(ii) X is ***connected*** if it is not disconnected.

Remark 6.3.2. If X is disconnected, then the U and V in question are also closed sets, since $U = X \setminus V$.

We immediately get the following characterisation:

Proposition 6.3.3. *A topological space X is connected if and only if the only subsets of X that are both open and closed are $\varnothing$ and X.*

Proof. Assume X is connected, and suppose there exists a set $U \neq \varnothing, X$ that is both open and closed. Then $V = X \setminus U$ is open and closed, and $X = U \cup V$, and by construction $U \cap V = \varnothing$. Hence X is disconnected, a contradiction.

Suppose X is disconnected. In other words, there exist open $U, V \neq \varnothing$ with $U \cap V = \varnothing$ and $X = U \cup V$. Then $U = X \setminus V$ is closed, being the complement of an open set. Hence U is both open and closed, a contradiction. □

Definition 6.3.4 (Connected subspace)**.** A subset of a topological space is ***connected*** if it is connected in the relative topology.

Example 6.3.5. Consider $\mathbb{Q} \subset \mathbb{R}$. Then $\mathbb{Q}$ is disconnected.

To see this, take any irrational number $a \in \mathbb{R} \setminus \mathbb{Q}$. Set $U = \mathbb{Q} \cap (a, \infty)$ and $V = \mathbb{Q} \cap (-\infty, a)$. Then U and V are open in $\mathbb{Q}$ by the definition of the relative topology, they don't meet, and $\mathbb{Q} = U \cup V$. ▲

Remark 6.3.6. The only connected subsets of $\mathbb{Q}$ are singletons, or one-point sets. Thus, $\mathbb{Q}$ is ***totally disconnected***.

Definition 6.3.7 (Totally disconnected space)**.** A topological space is ***totally disconnected*** if the only connected subsets are one-point sets.

Exercise 6.3. Show that any countable metric space is totally disconnected. ■

Theorem 6.3.8. *Let $f\colon X \to Y$ be a continuous function. Suppose X is connected. Then $f(X)$ is connected.*

Proof. Let $E \subset f(X)$ be both open and closed. We need to show $E = \varnothing$ or $E = f(X)$.

Since f is continuous, $f^{-1}(E)$ is open and closed in X. Since X is connected, $f^{-1}(E) = \varnothing$ or $f^{-1}(E) = X$, and we are done. □

Theorem 6.3.9. *Let $\{ E_\alpha \}_{\alpha \in \Lambda}$ be a family of connected subsets of X. Assume $E_\alpha \cap E_\beta \neq \varnothing$ for all $\alpha, \beta \in \Lambda$. Then*

$$\bigcup_{\alpha \in \Lambda} E_\alpha$$

is connected.

Proof. Let

$$E = \bigcup_{\alpha \in \Lambda} E_\alpha$$

and let $F \neq \varnothing$ be a subset of E that is both open and closed. We need to show that $F = E$, which is equivalent to $E_\alpha \subset F$ for all α.

Let $x \in F$. Then $x \in E_{\alpha_0}$ for some $\alpha_0 \in \Lambda$ since $F \subset E$.

Now $F \cap E_{\alpha_0} \neq \varnothing$, since they meet in at least x, and this set is both open and closed in E_{α_0}. But E_{α_0} is by assumption connected, meaning $F \cap E_{\alpha_0} = E_{\alpha_0}$ is the only option. In other words, $E_{\alpha_0} \subset F$.

For any $\beta \in \Lambda$, $F \cap E_\beta$ is open and closed in E_β. But this can't be empty, since $E_{\alpha_0} \subset F$. I.e.,

$$(E_{\alpha_0} \cap E_\beta) \subset (F \cap E_\beta)$$

implies $F \cap E_\beta \neq \varnothing$, so $F \cap E_\beta = E_\beta$. Hence $E_\beta \subset F$ for all $\beta \in \Lambda$. Thus $E = F$. □

Definition 6.3.10 (Connected component)**.** Let X be a topological space and $x \in X$. The ***connected component*** of x, denoted $C(x)$, is the union of all connected subsets that contain x.

From Theorem 6.3.9 we get immediately:

Corollary 6.3.11. *Two connected components of X either coincide or are disjoint.*

Exercise 6.4. A connected component of X is always closed. ■

Lecture 7 Different kinds of connectedness

7.1 More connectedness

Theorem 7.1.1. *(i) A finite product of connected spaces is connected.*

(ii) Let $\{ X_\alpha \}_{\alpha \in \Lambda}$ be a family of connected spaces. Then

$$X := \prod_{\alpha \in \Lambda} X_\alpha$$

is connected. (By convention, this product has the product topology.)

Proof. (i) It suffices to consider the product of two connected spaces X and Y. The result then follows from induction.

Fix $y_0 \in Y$. For $x \in X$, let

$$T_x(X \times \{ y_0 \}) \cup (\{ x \} \times Y).$$

Then $X \times \{ y_0 \}$ and $\{ x \} \times Y$ are connected since X and Y are (they're homeomorphic, we've just added a single point), and

$$(X \times \{ y_0 \}) \cap (\{ x \} \times Y) = \{ (x, y_0) \} \neq \varnothing.$$

By Theorem 6.3.9, this implies T_x is connected. Notice how

$$X \times Y = \bigcup_{x \in X} T_x$$

and for any $x_1, x_2 \in X$, $T_{x_1} \cap T_{x_2} \neq \varnothing$ (they meet in $X \times \{ y_0 \}$). Hence by Theorem 6.3.9 $X \times Y$ is connected.

(ii) First fix a base point $b = (b_\alpha)_{\alpha \in \Lambda} \in X$.

For any finite subset $\{ \alpha_1, \alpha_2, \dots, \alpha_n \} \subset \Lambda$, define

$$X \supset X(\alpha_1, \alpha_2, \dots, \alpha_n) = \{ (x_\alpha)_{\alpha \in \Lambda} \mid x_\alpha = b_\alpha \text{ for } \alpha \neq \alpha_1, \alpha_2, \dots, \alpha_n \}.$$

So, up to order of multiplication, this looks like

$$X_{\alpha_1} \times X_{\alpha_2} \times \cdots \times X_{\alpha_n} \times \{ b_\alpha \}_{\alpha \neq \alpha_i}.$$

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Thus $X(\alpha_1, \ldots, \alpha_n)$ is connected.

Let

$$Y \coloneqq \bigcup X(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where the union runs over finite subsets $\{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \subset \Lambda$.

Notice how $b \in X(\alpha_1, \alpha_2, \ldots, \alpha_n)$, so any $X(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap X(\beta, \beta, \ldots, \beta) \neq \varnothing$. Thus by the previous part Y is connected.

Notice how up until this point we have not used the topology of X. We claim $\overline{Y} = X$. The result then follows from the fact that if A if connected, then $\overline{A}$ is connected.

For any $x = (x_\alpha)_{\alpha \in \Lambda} \in X$ and any neighbourhood U of x, we need to show $U \cap Y \neq \varnothing$.

We can assume $U = \prod\limits_{\alpha \in \Lambda} U_\alpha$ is a basis element. I.e., $U_\alpha = X_\alpha$ for $\alpha \notin \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$.

Take $y = (y_\alpha)$ by

$$y_\alpha = \begin{cases} x_\alpha & \text{for } \alpha \in \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \\ b_\alpha & \text{for } \alpha \notin \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}. \end{cases}$$

Then $y = (y_\alpha) \in X(\alpha_1, \ldots, \alpha_n) \subset Y$ and also $y \in U$. Thus $Y \cap U \neq \varnothing$, and we are done. □

Exercise 7.1. Show that if A is connected, then $\overline{A}$ is connected. ■

Remark 7.1.2. The result does not necessarily hold if X is endowed with the box topology instead of the product topology.

Exercise 7.2. The only connected subsets of $\mathbb{R}$ are $\varnothing$, single-point sets, and (finite or infinite) intervals (open, closed, or half-open). ■

This, together with Theorem 6.3.8 saying that a continuous function sends connected sets to connected sets, gives us a slick proof of a famous Calculus result:

Theorem 7.1.3 (Intermediate value theorem)**.** *Let X be connected and $f \colon X \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ for some $a, b \in X$, then for any $r \in \mathbb{R}$ with $f(a) < r < f(b)$ there exists $c \in X$ such that $f(c) = r$.*

7.2 Path connectedness

Definition 7.2.1 (Path, path connected)**.** Let X be a topological space. Let $x, y \in X$.

(*i*) A ***path*** in X from x to y is a continuous function $r \colon [0, 1] \to X$ such that $r(0) = x$ and $r(1) = y$.

(*ii*) X is ***path-connected*** if for any $x, y \in X$, there exists a path from x to y.

Theorem 7.2.2. *A path-connected space is connected.*

Proof. Fix $x_0 \in X$. For each $x \in X$, let $r_x\colon [0,1] \to X$ be a path from x_0 to x. Since r_x is continuous and $[0,1]$ is connected, the path traced by $r_x([0,1])$ is connected. Likewise, a path r_y from x_0 to $y \in X$ gives a connected set $r_y([0,1])$.

Then

$$X = \bigcup_{x \in X} r_x([0,1])$$

is connected. □

Remark 7.2.3. Connected does not imply path connected.

Counterexample 7.2.4. Let $E = \left\{ (0,y) \middle| -1 \leq y \leq 1 \right\}$ and $F = \left\{ (x, \sin \frac{1}{y}) \middle| 0 < x \leq 1 \right\}$. Take $X = E \cup F = \overline{F}$.

Then X is connected (because it is the closure of a connected set) but it is not path-connected. ▲

Exercise 7.3. Verify that this set (known as the ***topologist's sine curve***) is not path-connected. ■

Remark 7.2.5. This gives an example showing that the closure of a path connected set is not necessarily path connected.

Definition 7.2.6 (Locally path-connected)**.** A space X is ***locally path-connected*** if, for each open subset V of X and each $x \in V$, there is a neighbourhood U of x such that x can be joined to any point of U by a path in V.

Exercise 7.4. Prove that the path components of a locally path-connected space coincide with the connected components. ■

Exercise 7.5. Show that an open subset of $\mathbb{R}^n$ is connected if and only if it is path-connected. ■

7.3 Compactness

Definition 7.3.1 (Compact)**.** A topological space X is ***compact*** if every open cover of X has a finite subcover.

In other words, for any family of open sets $\left\{ U_\alpha \right\}_{\alpha \in \Lambda}$ with

$$X = \bigcup_{\alpha \in \Lambda} U_\alpha$$

there exits $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that

$$X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Definition 7.3.2 (Compact subset)**.** A subset $E \subset X$ is ***compact*** if E is compact in the relative topology.

Proposition 7.3.3. *Any finite union of compact sets is compact.*

Exercise 7.6. Prove Proposition 7.3.3. ■

Theorem 7.3.4. *A closed subspace of a compact topological space is compact.*

Proof. Let X be compact and $E \subset X$ closed. For any open cover $\{ U_\alpha \}_{\alpha \in \Lambda}$ of E,

$$(X \setminus E) \cup \bigcup_{\alpha \in \Lambda} U_\alpha$$

is an open cover of X (since $X \setminus E$ is open because E is closed). Since X is compact, there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$(X \setminus E) \cup \bigcup_{i=1}^{n} U_\alpha$$

is an open cover of X. Hence $\{ U_{\alpha_i} \}_{i=1}^{n}$ is an open cover of E. □

Lemma 7.3.5. *Let X be a Hausdorff space. Let $E \subset X$ be a compact subset. Then for each $x \in X \setminus E$, there exists disjoint open sets U and V such that $x \in U$ and $E \subset V$.*

Proof. Let $x \in X \setminus E$ be fixed. For each $y \in E$, there exists disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$ since X is Hausdorff.

Then $\{ V_y \}_{y \in E}$ is an open cover of E. Since E is compact there exists a finite open subcover, say $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$.

Then taking

$$V \coloneqq V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$$

we have $E \subset V$ and taking

$$U \coloneqq U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}.$$

This is open since it is a finite intersection of open sets, and by construction $V \cap U = \varnothing$. □

Theorem 7.3.6. *A compact subset in a Hausdorff space is closed.*

Proof. Let X be Hausdorff and $E \subset X$ compact. By the previous lemma, for each $x \in X \setminus E$, there exists an open U with $x \in U$ and $U \in X \setminus E$. Hence $X \setminus E$ is open, and E is closed. □

Exercise 7.7. Let X be a Hausdorff space. Suppose A and B are disjoint compact subsets of X. Show that there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. ■

Theorem 7.3.7. *Let X be a compact space and Y be any topological space. Suppose $f \colon X \to Y$ is continuous. Then $f(X)$ is a compact subset of Y.*

Proof. Let $\{ U_\alpha \}_{\alpha \in \Lambda}$ be an open cover of $f(X)$. Then

$$\{ f^{-1(U_\alpha)} \}_{\alpha \in \Lambda}$$

is an open cover of X. Since X is compact, we can find a finite subcover, say

$$X = f^{-1}(U_{\alpha_1}) \cup f^{-1}(U_{\alpha_2}) \cup \cdots \cup f^{-1}(U_{\alpha_n}).$$

Mapping back,

$$f(X) \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}.$$

Hence $f(X)$ is compact. □

Like with the Intermediate value theorem above, this gives a slick proof of the Extreme value theorem in analysis.

Recall another theorem from analysis:

Theorem 7.3.8 (Heine–Borel theorem)**.** *Let $E \subset \mathbb{R}^n$. Then E is compact if and only if E is closed and bounded.*

Remark 7.3.9. Let X be a metric space. Then $E \subset X$ implies E is closed and bounded (closed because metric spaces are Hausdorff).

The converse is not true in general, though it is true for any finite-dimensional vector space.

Lecture 8 Tychonoff's theorem

8.1 Open maps

Theorem 8.1.1. *Let X be a compact space and Y be a Hausdorff space. Suppose $f: X \to Y$ is continuous. If f is one-to-one, then f is a homeomorphism of X and $f(X)$.*

Proof. It suffices to show f is an open mapping, i.e., f sends open sets to open sets, or $f(U)$ is open in $f(X)$ for all open $U \subset X$.

Note that $X \setminus U$ is closed, being the complement of an open set. Since X is compact, $X \setminus U$ is also compact. Thus $f(X \setminus U) = f(X) \setminus f(U)$ because f is one-to-one. This image is compact in $f(X) \subset Y$ by Theorem 7.3.7. Since Y is Hausdorff, $f(X)$ must be closed by Theorem 7.3.6. Hence $f(X) \setminus f(U)$ is closed in $f(X)$, so $f(U)$ is open in $f(X)$. □

8.2 The Tychonoff theorem, finite case

We want to prove that an arbitrary product of compact spaces is compact.

There are two natural cases: the finite product case and the infinite product case. We start with the finite product case.

Lemma 8.2.1. *Let X be a topological space and let $\mathscr{B}$ be a basis for the topology of X. If every open cover of X by sets in $\mathscr{B}$ has a finite subcover, then X is compact.*

Note that compact means any cover has a finite subcover. This lemma says that we really only need to consider the basis elements.

Proof. Let $\{ U_\alpha \}_{\alpha\in\Lambda}$ be an open cover of X. For each $x \in X$, $x \in U_{\alpha_x}$ for some $\alpha_x \in \Lambda$. Since U_{α_x} we can choose $V_x \in \mathscr{B}$ such that $x \in V_x \subset U_{\alpha_x}$. Then $\{ V_x \}_{x\in X} \subset \mathscr{B}$ is an open cover of X.

By assumption there exist $x_1, x_2, \dots, x_n \in X$ such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_n},$$

but each $V_{x_i} \subset U_{\alpha_i}$, so

$$X = U_{\alpha_{x_1}} \cup U_{\alpha_{x_2}} \cup \cdots \cup U_{\alpha_{x_n}}$$

is a finite subcover from the original family. □

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Theorem 8.2.2 (Tychonoff's theorem, finite case). *If $X_1, X_2, \dots, X_n$ are compact spaces, then $X_1 \times X_2 \times \cdots \times X_n$ is compact.*

Proof. It suffices to consider the case $n = 2$. The theorem follows by induction.

Let $X = X_1 \times X_2$. Let

$$\mathscr{C} = \left\{ U_\alpha \times V_\alpha \right\}_{\alpha \in \Lambda}$$

be an open cover of X with U_α open in X_1 and V_α open in X_2.

By the above Lemma it suffices to show that $\mathscr{C}$ has a finite subcover.

For any fixed $y \in X_2$, $X_1 \times \left\{ y \right\}$ is compact (since it is homeomorphic to X_1). Hence there exists a finite cover of $X_1 \times \left\{ y \right\}$ in $\mathscr{C}$, say

$$U_{y,1} \times V_{y,1}, \quad U_{y,2} \times V_{y,2}, \quad \dots \quad U_{y,n} \times V_{y,n}.$$

Then

$$V_y \coloneqq V_{y,1} \cap V_{y,2} \cap \cdots \cap V_{y,n}$$

is an open neighbourhood of y in X_2 (it's a finite intersection of open sets).

Note that for $\pi_2 \colon X_1 \times X_2 \to X_2$ defined by $\pi_2(x, y) = y$, $\pi_2^{-1}(V_y)$ is covered by

$$U_{y,1} \times V_{y,1}, \quad U_{y,2} \times V_{y,2}, \quad \dots \quad U_{y,n} \times V_{y,n}.$$

Since $\left\{ V_y \right\}_{y \in X_2}$ is an open cover of X_2 and X_2 is compact,

$$X_2 = V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$$

for some finite collection $y_1, y_2, \dots, y_n \in X_2$. Now

$$X = \pi_2^{-1}(V_{y_1}) \cup \pi_2^{-1}(V_{y_2}) \cup \cdots \cup \pi_2^{-1}(V_{y_n})$$

and each $\pi_2^{-1}(V_{y_i})$ is covered by finitely many sets in $\mathscr{C}$.

Hence X is covered by finitely many sets in $\mathscr{C}$, finishing the proof. □

8.3 Axiom of choice

In order to prove the infinite case of Tychonoff's theorem we have to talk about:

Axiom 8.3.1 (Axiom of choice). Given a family of nonempty sets, it is possible to select precisely one element from each member of the family.

In other words, if $\left\{ S_\alpha \right\}_{\alpha \in \Lambda}$ is a family of nonempty sets indexed by Λ, then there exists a function f on Λ such that $f(\alpha) \in S_\alpha$ for all $\alpha \in \Lambda$.

This seems very natural, but it can't be proved from basic set theory axioms. It also has some seemingly unnatural consequences, like the Banach–Tarsky theorem.

8.4 Zorn's lemma

Definition 8.4.1 (Partially ordered set). A ***partially ordered set*** is a nonempty set S with a relation "$\leq$" satisfying

(*i*) $x \leq x$ for all $x \in S$ (reflexivity);

(*ii*) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity); and

(*iii*) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).

Note that this does *not* say that any two elements can be compared, only that if they can be compared, the above holds.

Definition 8.4.2 (Totally ordered set)**.** Let S be a partially ordered set. Let $E \subset S$ be a subset.

(*i*) E is ***totally ordered*** if for any $x, y \in E$ we have $x \leq y$ or $y \leq x$.

(*ii*) An ***upper bound*** for E is an element $x \in S$ such that $y \leq x$ for all $y \in E$.

(*iii*) An element $x \in E$ is a ***maximal element*** of E if $x \leq y$ for some $y \in E$, then $y = x$.

Remark 8.4.3. The distinction between upper bound and maximal element is that the upper bound only needs to come from S; there could be two upper bounds of E in S that are not mutually comparable.

Lemma 8.4.4 (Zorn's lemma)**.** *Let $S \neq \varnothing$ be a partially ordered set. Suppose each totally ordered subset of S has an upper bound. Then S has a maximal element.*

Remark 8.4.5. The Axiom of choice and Zorn's lemma are equivalent.

Unfortunately, the infinite product case of Tychonoff's theorem is also equivalent to the Axiom of choice. So what we will do is assume Zorn's lemma and show that Tychonoff's theorem follows.

But first:

Theorem 8.4.6 (Alexander subbasis theorem)**.** *Let X be a topological space and let $\mathscr{B}$ be a subbasis for the topology of X.*

If every open cover of X by sets in $\mathscr{B}$ has a finite subcover, then X is compact.

Proof. Suppose X is not compact. In other words, there exists an open cover $\mathscr{C}$ of X such that no finite subfamily of $\mathscr{C}$ covers X.

We will show that there exists a cover of X by sets in $\mathscr{B}$ that has no finite subcover of X, which is a contradiction.

Consider families $\mathscr{D}$ of open subsets in X such that $\mathscr{C} \subset \mathscr{D}$ and no finite collection of sets in $\mathscr{D}$ covers X.

Set $\mathscr{P} = \left\{ \mathscr{D} \right\}$ ordered by inclusion. Then $\mathscr{P} \neq \varnothing$ (since $\mathscr{C} \in \mathscr{P}$), and $\mathscr{P}$ is partially ordered.

Let $\left\{ \mathscr{D}_\alpha \right\}_{\alpha \in \Lambda}$ be a totally ordered subset of $\mathscr{P}$. Set

$$\mathscr{D} = \bigcup_{\alpha \in \Lambda} \mathscr{D}_\alpha.$$

Clearly $\mathscr{D}$ is an upper bound for $\left\{ \mathscr{D}_\alpha \right\}$; we claim that $\mathscr{D} \in \mathscr{P}$.

I.e., no finite collection of sets in $\mathscr{D}$ covers X.

To see this, suppose there exists a finite collection of sets in $\mathscr{D}$ that covers X, say

$$U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n},$$

from $\mathscr{D}_{\alpha_1} \subset \mathscr{D}_{\alpha_2} \subset \cdots \subset \mathscr{D}_{\alpha_n}$. Then these sets must belong to $\mathscr{D}_\alpha$ for some $\alpha \in \Lambda$, which contradicts $\mathscr{D}_\alpha$ having no such finite subcover.

By Zorn's lemma, $\mathscr{P}$ therefore has a maximal element, say $\mathscr{E}$.

Finally we claim that $\mathscr{E} \cap \mathscr{B}$ covers X. If true, since $\mathscr{E} \subset \mathscr{P}$ says this is a cover of X by sets in $\mathscr{B}$ that has no finite subcover. This is a contradiction.

Let $x \in X$. It suffices to show that there exists $V \in \mathscr{E} \cap \mathscr{B}$ such that $x \in V$.

Choose $U \in \mathscr{E}$ such that $x \in U$. Since $\mathscr{B}$ is a subbasis there exists $V_1, V_2, \ldots, V_m \in \mathscr{B}$ such that

$$x \in V_1 \cap V_2 \cap \cdots \cap V_m \subset U.$$

If $V_j \in \mathscr{E}$ for some j, then we are done since $x \in V_j \in \mathscr{E} \cap \mathscr{B}$.

If no $V_j \in \mathscr{E}$ for all j, we will derive a contradiction. We have $\mathscr{E} \cup \{ V_j \} \notin \mathscr{P}$ for $j = 1, 2, \ldots, m$ since $\mathscr{E}$ is maximal in $\mathscr{P}$. This means for each j, there exists finitely many open sets $W_{j,1}, \ldots, W_{j,n_j} \in \mathscr{E}$ such that

$$X = V_j \cup \Big(\bigcup_{i=1}^{n_j} W_{j,i} \Big).$$

Take intersections over j, so

$$X = \bigcap_{j=1}^{m} \Big(V_j \cup \Big(\bigcup_{i=1}^{n_j} W_{j,i} \Big) \Big) \subset (V_1 \cap V_2 \cap \cdots \cap V_m) \cup \Big(\bigcup_{i,j} W_{i,j} \Big).$$

But $V_1 \cap V_2 \cap \cdots \cap V_m \subset U$ by choice of U, so

$$X = U \cup \Big(\bigcup_{i,j} W_{i,j} \Big),$$

but all of these are in $\mathscr{E}$, so $\mathscr{E}$ has a finite subcover of X. This contradicts $\mathscr{E} \in \mathscr{P}$. ☐

Lecture 9 Tychonoff's theorem

9.1 The infinite case

We will now use the Alexander subbasis theorem in order to prove:

Theorem 9.1.1 (Tychonoff's theorem, infinite case)**.** *Let* $\{ X_\alpha \}_{\alpha \in \Lambda}$ *be a family of compact spaces. Then*

$$X = \prod_{\alpha \in \Lambda} X_\alpha$$

is compact (in the product topology).

Proof. By the Alexander subbasis theorem, X is compact if it satisfies the following property: Let $\mathscr{B}$ be a subbasis for the topology on X. If for any family $\mathscr{D}$ of subsets in $\mathscr{B}$, $\mathscr{D}$ has no finite subcover of X, then $\mathscr{D}$ does *not* cover X.

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Now take $\mathscr{B}$ to be the subbasis for the product topology of X of the form $\pi_\alpha^{-1}(U_\alpha)$, $\alpha \in \Lambda$, where U_α is open in X_α. These pullbacks look like $U_\alpha \times \{ X_\beta \}_{\beta \neq \alpha}$.

Let $\mathscr{D}$ be a family of subsets such that $\mathscr{D} \subset \mathscr{B}$ and $\mathscr{D}$ has no finite subcover of X.

We want to show that $\mathscr{D}$ does not cover X.

For any $\beta \in \Lambda$, consider the open subsets $V \subset X_\beta$ such that $\pi_\beta^{-1}(V) \in \mathscr{D}$.

The family of such sets V cannot cover X_β. Otherwise, since X_β is compact, we would have a finite subcover of X_β, say $V_1, V_2, \ldots, V_n$, for which $\pi_\beta^{-1}(V_i) \in \mathscr{D}$. But those pullbacks would cover X, and $\mathscr{D}$ has no finite subcover of X.

Hence we can choose a point

$$x_\beta \in X_\beta \setminus \{ V \mid \pi_\beta^{-1}(V) \in \mathscr{D} \text{ and } V \text{ open in } X_\beta \}.$$

(If no such V exist, we are just subtracting the empty set.)

Then

$$x = (x_\beta)_{\beta \in \Lambda} \in X$$

is not included in any sets in $\mathscr{D}$ by construction.

Hence $\mathscr{D}$ does not cover X since it misses at least x. □

Remark 9.1.2. Note that the existence of this x is by Axiom of choice.

In fact, the infinite case of Tychonoff's theorem is equivalent to the Axiom of choice.

Exercise 9.1. Show that Tychonoff's theorem implies the Axiom of choice. ■

9.2 Other notions of compactness

Recall that compact means every open cover has a finite subcover. There are other types of compactness:

Definition 9.2.1 (Limit point compactness)**.** A space X is ***limit point compact*** if every infinite subset of X has a limit point.

Definition 9.2.2 (Sequential compactness)**.** A space X is ***sequentially compact*** if every sequence in X has a convergent subsequence.

Note that this last notion, in $\mathbb{R}^n$, is usually discussed in terms of the ***Bolzano–Weierstrass theorem***, saying that any bounded sequence has a convergent subsequence (since a bounded sequence is contained in a compact set).

Note that in $\mathbb{R}^n$, all of these notions are equivalent. We will show that in fact in any metrisable space, all three notions are equivalent.

Recall that $x \in X$ is a ***limit point*** of a subset $A \subset X$ if for every open neighbourhood U of x, we have $U \cap (A \setminus \{ x \}) \neq \varnothing$. In other words, $x \in \overline{A \setminus \{ x \}}$.

Theorem 9.2.3. *If X is compact, then X is limit point compact.*

Note that we did not assume X is metrisable here—compactness is stronger than limit point compactness.

Proof. We need to show that if $A \subset X$ is infinite, then A has a limit point. This is equivalent to showing that if A has no limit point, then A is a finite set.

Assume A has no limit point.

We claim A is closed, i.e. $X \setminus A$ is open.

For any $x \in X \setminus A$, x is not a limit point of A. Hence $x \notin \overline{A \setminus \{ x \}} = \overline{A}$. This means $x \in X \setminus \overline{A}$ which is open, so there exists an open neighbourhood U of x such that $U \subset X \setminus \overline{A}$. Therefore $x \in U \subset (X \setminus \overline{A}) \subset X \setminus A$. Hence $X \setminus A$ is open, so A is closed.

Our goal is to show that A is finite. To this end, notice how for each $x \in A$, since x is not a limit point of A, there exists an open neighbourhood U_x of x such that $U_x \cap (A \setminus \{ x \}) = \varnothing$. This means $U_x \cap A = \{ x \}$, i.e., every point in A is an isolated point.

Now consider

$$\mathscr{C} = \{ X \setminus A \} \cup \{ U_x \mid x \in A \}.$$

This is an open cover of X, and X is compact so there exists a finite subcover, say

$$X \setminus A, U_{x_1}, U_{x_2}, \ldots, U_{x_n}.$$

Thus

$$X = (X \setminus A) \cup (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}).$$

In particular this means the union in the parenthesis covers A, so

$$A = A \cap (U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}),$$

and each of those intersections is just x_i, so

$$A = \{ x_1, x_2, \ldots, x_n \}$$

is a finite set. □

Remark 9.2.4. In general, limit point compactness does *not* imply compactness.

Counterexample 9.2.5. Let $Y = \{ a, b \}$ be a two point set with the trivial topology, i.e., $\mathscr{T} = \{ \varnothing, Y \}$. Let $X = \mathbb{Z}_+ \times Y$ where

$$\mathbb{Z}_+ = \{ n > 0 \mid n \in \mathbb{Z} \}$$

with the discrete topology (so every single point is an open set).

Then X is limit point compact. In fact, for any nonempty $A \subset X$, A has a limit point. If $(n, a) \in A$, then (n, b) is a limit point of A.

But X is not compact. Take $\mathscr{C} = \{ U_n = \{ n \} \times T \mid n \in \mathbb{Z}_+ \}$. This is an open cover of X, but it has no finite subcover since removing any set makes it miss a point of $\mathbb{Z}_+$. ▲

Theorem 9.2.6. *Let X be a metrisable space. Then the following are equivalent:*

(*i*) *X is compact.*

(*ii*) *X is limit point compact.*

(*iii*) *X is sequentially compact.*

Proof. That (i) implies (ii) is true in general: this is Theorem 9.2.3.

To show (ii) implies (iii), let $\{ x_n \}$ be a sequence in X. Consider the set $A = \{ x_n \mid n \in \mathbb{Z}_+ \}$.

First, if A is a finite set, then x_n must repeat at least one element infinitely many times. I.e., there exists some $x \in X$ such that $x = x_n$ for infinitely many n. This means $\{ x_n \}$ has a subsequence $\{ x_{n_k} = x \}_{n_k}$ which is a constant sequence, hence convergent (to x).

Second, if A is an infinite set, then since X is limit point compact A must have a limit point, say x. We select a subsequence of $\{ x_n \}$ that converges to x as follows.

For any $\delta > 0$, $B(x, \delta) \cap A$ has infinitely many points since x is a limit point, because otherwise there exists some sufficiently small $\delta' > 0$ such that $B(x, \delta') \cap A = \varnothing$, contradicting x being a limit point of A.

Thus we can choose $n_1 < n_2 < n_3 < \dots$ such that $x_{n_i} \in B(x, \frac{1}{i}) \cap A$. Then $\{ x_{n_i} \}_{i=1}^{\infty}$ converges to x.

(Note that this is really the same argument one uses to prove the Bolzano–Weierstrass theorem in $\mathbb{R}^n$.)

Finally, (iii) implies (i). Assume X is sequentially compact with a metric d.

First, an important lemma:

Lemma 9.2.7 (Lebesgue number lemma)**.** *Let X be a metrisable space and suppose X is (sequentially) compact.*

For any open cover $\mathscr{C}$ of X there exists $\delta > 0$ (depending on $\mathscr{C}$) such that for each subset V of X with diameter $\operatorname{diam}(V) < \delta$*, there exists $U \in \mathscr{C}$ such that $V \subset U$.*

Note that this is uniform in V; it only depends on $\mathscr{C}$. Here

Definition 9.2.8 (Diameter)**.** The ***diameter*** of a subset V in a metric space is defined by

$$\operatorname{diam}(V) = \sup\{ d(x_1, x_2) \mid x_1, x_2 \in V \}.$$

We prove this by contradiction: Assume there is no $\delta > 0$ such that $V \subset X$ with $\operatorname{diam}(V) < \delta$ that has an element of $\mathscr{C}$ containing it.

Then for each $n \in \mathbb{N}$ there exists $V_n \subset X$ such that $\operatorname{diam}(V_n) < \frac{1}{n}$ and V_n is not contained in any element of $\mathscr{C}$.

Choose $x_n \in V_n$ for each n; this sequence must have a convergent subsequence, say $\{ x_{n_i} \}$, since X is sequentially compact.

Assume x_{n_i} converges to $x \in X$. Since $\mathscr{C}$ covers X, $x \in V$ for some $V \in \mathscr{C}$. Since V is open, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V$. Choose n_i large enough such that $\frac{1}{n_i} < \frac{\varepsilon}{2}$ and $d(x_{n_i}, x) < \frac{\varepsilon}{2}$.

Then for any $y \in V_{n_i}$,

$$d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) < \frac{1}{n_i} + \frac{\varepsilon}{2} < \varepsilon.$$

Hence $y \in B(x, \varepsilon) \subset V$. I.e., $V_{n_i} \subset V \in \mathscr{C}$ contradicts no element of $\mathscr{C}$ containing any V_n.

Lecture 10 Compactnesses

10.1 Proof continued

Proof of Theorem 9.2.6, continued. We are left with (iii) implying (i), for which we showed the Lebesgue number lemma.

Next, if X is sequentially compact, then given any $\varepsilon > 0$, there exists a finite cover of X by ε-balls (in other words, X is ***totally bounded***).

We show this by contradiction. Assume there exists some $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls.

Choose $x_1 \in X$, then $x_2 \in X \setminus B(x_1, \varepsilon)$, $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$, and so on. The sets we are removing are nonempty by the assumption of ε-balls not covering X.

Hence we get a sequence $\{ x_n \}$ in X that does not have a convergent subsequence, since $d(x_{n+1}, x_i) \geq \varepsilon$ for $i = 1, 2, \ldots, n$. This contradicts X being sequentially compact.

Third: we are not equipped to show that if X is sequentially compact, then X is compact.

Let $\mathscr{C}$ be an open cover of X. By Lebesgue number lemma, there exists some $\delta > 0$ such that for any $V \subset X$ with $\operatorname{diam}(V) < \delta$, we have $V \subset U$ for some $U \in \mathscr{C}$.

Take $\varepsilon = \frac{\delta}{3}$. By the second step, we know X can be covered by finitely many ε-balls. In other words,

$$X = B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon).$$

Now $\operatorname{diam}(B(x_i, \varepsilon)) < \delta$, so $B(x_i, \varepsilon) \subset U_i$ for some $U_i \in \mathscr{C}$. Hence from the covering of balls we get a finite cover

$$X = U_1 \cup U_2 \cup \cdots \cup U_n$$

of X. □

Remark 10.1.1. There exist sequentially compact spaces that are not compact.

There also exist compact spaces that are not sequentially compact.

For posterity we write down the definition from inside the proof:

Definition 10.1.2 (Totally bounded)**.** A metric space X is ***totally bounded*** if for any $\varepsilon > 0$ there is a finite cover of X by ε-balls.

In other words, we showed in the second step of the Theorem 9.2.6 that

Corollary 10.1.3. *If X is a compact metric space, then X is totally bounded.*

The converse is not true: we will show later that if X is a metric space, X is compact if and only if X is complete and totally bounded.

Exercise 10.1. Let X be a metric space.

(a) Show that if E is a compact subset of X, then E is closed and bounded.

(b) Give an example to show the converse is false. In other words, find a metric space in which not every closed and bounded subset if compact. ■

Date: September 29th, 2020.

10.2 Uniform continuity

Definition 10.2.1 (Uniform continuity)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f\colon X \to Y$ is ***uniformly continuous*** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \varepsilon$.

Theorem 10.2.2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ be continuous. Assume X is compact. Then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$, for each $y \in Y$ let $V_y = B(y, \frac{\varepsilon}{2}) \subset Y$. Since f is continuous,

$$\mathscr{C} = \left\{\, f^{-1}(V_y) \;\middle|\; y \in Y \,\right\}$$

is an open cover of X. Since X is compact, by Lebesgue number lemma there exists $\delta > 0$ such that for any $U \subset X$ with $\operatorname{diam}(U) \le 2\delta$, U is contained in some element of $\mathscr{C}$.

Hence for any $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $x_2 \in B(x_1, \delta)$. Since the diameter of this is bounded by 2δ, this is contained in some $f^{-1}(V_y)$.

Hence $f(x_1), f(x_2) \in V_y = B(y, \frac{\varepsilon}{2})$, so $d_Y(f(x_1), f(x_2)) < \varepsilon$. □

Exercise 10.2. Let X and Y be two topological spaces. Assume that Y is compact. Let N be an open set of $X \times Y$ containing $\{\, x_0 \,\} \times Y$ for some $x_0 \in X$. Show that there exists an open neighbourhood W of x_0 in X such that $(W \times Y) \subset N$. ■

Exercise 10.3. Let X and Y be two topological spaces.

(a) Assume that Y is compact. Show that the projection map $\pi_1\colon X \times Y \to X$ is a closed map. That is, for any closed subset E of $X \times Y$, $\pi_1(E)$ is closed.

(b) Is (a) still true if we do not assume Y is compact? ■

Exercise 10.4. Let X and Y be two topological spaces. Let $f\colon X \to Y$ be a map. Let

$$G_f = \left\{\, (x, f(x)) \in X \times Y \;\middle|\; x \in X \,\right\}$$

be the graph of f.

(a) Assume that Y is compact Hausdorff. Show that f is continuous if and only if G_f is closed in $X \times Y$.

(b) Is (a) true if Y is Hausdorff but *not* compact? ■

Exercise 10.5. Let (X, d) be a compact metric space. Let $f\colon X \to X$ be a (continuous) map. Let $0 < \alpha < 1$ be a real number. Suppose that

$$d(f(x), f(y)) \le \alpha d(x, y)$$

for all $x, y \in X$. Show that there exists a unique point $x \in X$ such that $f(x) = x$.

This is known as the ***Banach fixed-point theorem***. ■

10.3 Local compactness

Definition 10.3.1 (Local compactness)**.** (i) A space X is ***locally compact*** at $x \in X$ if there is an open neighbourhood of x such that $\overline{U}$ is compact.

(ii) A space X is ***locally compact*** if X is locally compact at every point $x \in X$.

Example 10.3.2. The space $\mathbb{R}^n$ is locally compact. (But it is not compact; it is unbounded.) ▲

Exercise 10.6. Let X be a Hausdorff space. Prove that X is locally compact at $x \in X$ if and only if for every open neighbourhood U of x, there is a neighbourhood V such that $\overline{V} \subset U$ and $\overline{V}$ is compact. ■

10.4 One-point compactification

Let X be a locally compact Hausdorff space which is not compact (e.g. $\mathbb{R}^n$).

Take a point, denoted by ∞, with $\infty \notin X$. Set $Y = X \cup \{ \infty \}$.

We define a topology (i.e. a collection of open sets) $\mathscr{T}$ on Y such that Y is compact. In particular, we have the open sets

(i) U where $U \subset X$ is open in X,

(ii) $Y \setminus C$ where $C \subset X$ is compact.

Example 10.4.1. The model for this is taking the number line $\mathbb{R}$, identifying both 'ends' of the number line as ∞, and gluing them together to form a circle. ▲

Exercise 10.7. Show that $\mathscr{T}$ as defined above is a topology on Y. ■

Theorem 10.4.2. *The space $Y = X \cup \{ \infty \}$ is a compact Hausdorff space, the relative topology for X inherited from Y coincides with the original topology on X, and $\overline{X} = Y$ in Y.*

Proof. First let us show that the two topologies coincide. Let $(X, \mathscr{C})$ be the original topology on X. Then we want to show that $(X, \mathscr{C}) = (X, \mathscr{T})$, and $\overline{X} = Y$.

For any open set U in Y, we have two options: either $U \subset X$, so $U \in \mathscr{C}$, in which case $U \cap X = U$, where the left-hand side is open in $\mathscr{T}$.

If $U = Y \setminus C$ where $C \subset X$ is compact, then $U \cap X = X \setminus C$. But X is Hausdorff and C is compact, so C is closed, meaning that $X \setminus C \in \mathscr{C}$. is open.

Hence $(X, \mathscr{T}) \subset (X, \mathscr{C})$.

On the other hand, for $U \in \mathscr{C}$, $U \in \mathscr{T}$. Hence $(X, \mathscr{C}) \subset (X, \mathscr{T})$.

Next, we show $\overline{X} = Y$. Since X is not compact, for any open set $Y \setminus C$ containing ∞, $(Y \setminus C) \cap X = X \setminus C \neq \varnothing$. Hence $\infty \in \overline{X}$.

Now let us show Y is compact. Let $\mathscr{A}$ be an open cover of Y. Then $Y \setminus C \in \mathscr{A}$ for some $C \subset X$ compact (since those are the neighbourhoods of ∞). Then

$$\mathscr{A} \setminus \{ Y \setminus C \}$$

is an open cover of C. But C is compact, so there is a finite subcover $\mathscr{A}' \subset \mathscr{A}$ of C. Then

$$\mathscr{A}' \cup \{ Y \setminus C \}$$

is a finite subcover of Y.

That Y is Hausdorff is easy. Let $x, y \in Y$. If $x, y \in X$, then since X is Hausdorff, there exist open $U, V \subset X$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

If $x \in X$ and $y = \infty$, since X is locally compact, there exists an open U such that $x \in U$ and $\overline{U}$ is compact. Then $y = \infty \in Y \setminus \overline{U}$, which is open in Y, and $(Y \setminus \overline{U}) \cap U = \varnothing$. ☐

Example 10.4.3. The one-point compactification $\mathbb{R} \cup \{ \infty \}$ is homeomorphic to $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$. ▲

Similarly,

Example 10.4.4. The one-point compactification $\mathbb{R}^n \cup \{ \infty \} \simeq S^n$, where

$$S^n = \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \}.$$

▲

Example 10.4.5. The one-point compactification $\mathbb{C} \cup \{ \infty \}$ is the ***Riemann sphere***. ▲

Example 10.4.6. Since $(0, 1)$ and $\mathbb{R}$ are homeomorphic, so must their one-point compactification be, i.e.

$$(0, 1) \cup \{ \infty \} \simeq \mathbb{R} \cup \{ \infty \} = S^1.$$

On the other hand, we can compactify $(0, 1)$ by adding both endpoints, i.e., $[0, 1]$. But these two spaces are *not* homeomorphic. ▲

Remark 10.4.7. There can be many compactifications of a space; the one-point compactification is just one method.

We will introduce another compactification: the Stone–Čech compactification, which is useful for analysis.

Lecture 11 Countability and separation axioms

11.1 Countability axioms

Definition 11.1.1 (First-countable)**.** A topological space is ***first-countable*** if for each $x \in X$ there exists a sequence of open neighbourhoods $\{ U_n \}_{n=1}^{\infty}$ of x such that each neighbourhood of x includes one of the U_n.

Exercise 11.1. (*a*) Any metric space is first-countable.

(*b*) Let X be first-countable and let $A \subset X$. Then $x \in \overline{A}$ (so x is a limit point of A) if and only if there is a sequence of points in A converging to x. ■

Definition 11.1.2 (Second-countable)**.** A topological space X is ***second-countable*** if X has a countable basis for its topology.

Exercise 11.2. A second-countable space is first-countable. ■

Date: October 1st, 2020.

Exercise 11.3. Let X be a compact Hausdorff space. Show that X is metrisable if and only if X is second-countable. ■

Remark 11.1.3. There exist metric spaces that are not second-countable. (Though they are not Euclidean spaces; there we can make a countable basis for the topology using rational intervals.)

Theorem 11.1.4 (Lindelöf's theorem)**.** *Let X be a second-countable topological space. Then every open cover of X has a countable subcover.*

Proof. Let $\mathscr{A}$ be an open cover of X. Let $\mathscr{B}$ be a countable basis for the topology on X.

Let $\mathscr{C} \subset \mathscr{B}$ consisting of sets $U \in \mathscr{B}$ such that $U \subset V$ for some $V \in \mathscr{A}$.

We claim that $\mathscr{C}$ is a cover of X.

This is easy: for each $x \in X$, $x \in V$ for some $V \in \mathscr{A}$ since $\mathscr{A}$ covers X. Since V is open and $\mathscr{B}$ is a basis, there must exist some $U \in \mathscr{B}$ such that $x \in U \subset V$. Hence $x \in U\mathscr{C}$.

Now $\mathscr{C} \subset \mathscr{B}$, and the latter is countable, so the former is countable. Write $\mathscr{C} = \{ U_n \}_{n=1}^{\infty}$, and for each U_n choose $V_n \in \mathscr{A}$ such that $U_n \subset V_n$. Then $\{ V_n \}_{n=1}^{\infty} \subset \mathscr{A}$ is a countable subcover of X. □

Definition 11.1.5 (Dense, separable)**.** (*i*) A subset $E \subset X$ is ***dense*** if $\overline{E} = X$.

(*ii*) A topological space X is ***separable*** if there is a countable subset of X that is dense in X.

Example 11.1.6. The real numbers $\mathbb{R}$ is separable, because $\mathbb{Q} \subset \mathbb{R}$ is dense; $\overline{\mathbb{Q}} = \mathbb{R}$. ▲

Theorem 11.1.7. *If X is second-countable, then X is separable.*

Proof. Let $\{ U_n \}_{n=1}^{\infty}$ be a basis for X. For each $U_n \neq \varnothing$, choose $x_n \in U_n$. Then $E = \{ x_n \}_{n=1}^{\infty}$ is a countable set which is dense in X.

To see this, pick an arbitrary $x \in X$. For any open neighbourhood U of x, we need to show that $U \cap E \neq \varnothing$, so that x is a limit point of E. But U is open, so there exists some U_n in the basis with $U_n \subset U$, and $x_n \in U_n$, so $x_n \in U \cap E \neq \varnothing$. □

Example 11.1.8. As one might show in measure theory or functional analysis classes, $L^p(X)$ is separable for $1 \leq p < \infty$. ▲

However $L^\infty(X)$ is *not* separable; this is the main difference between L^p and L^∞.

Exercise 11.4. Show the above; that $L^p(X)$ is separable for $1 \leq p < \infty$, but not for $p = \infty$. ■

11.2 Separation axiom

These are denoted with T because 'separation axiom' is 'Trennungsaxiom' in German.

Definition 11.2.1 (T_1-space)**.** A space X is a T_1***-space*** if for disjoint $x, y \in X$, there exists an open set U such that $x \notin U$ and $y \in U$.

Proposition 11.2.2. *X is a T_1-space if and only if single-point sets are closed.*

Proof. Forwards, fix $x \in X$. For each $y \in X \setminus \{ x \}$, there exists an open U_y such that $x \notin U_y$. I.e., $U_y \subset X \setminus \{ x \}$, and hence

$$X \setminus \{ x \} = \bigcup_{y \in X \setminus \{ x \}} U_y$$

is open.

Conversely, for distinct $x, y \in X$, $X \setminus \{ x \}$ is open since $\{ x \}$ is closed. Hence there exists an open U such that $y \in U$ and $x \notin U$. □

Exercise 11.5. Let X be a topological space and let X_0 be the topological space made up of the set X with the cofinite topology. Show that the identity map from X to X_0 is continuous if and only if X is a T_1-space. ■

Definition 11.2.3 (T_2-space, Hausdorff)**.** A space X is a ***T_2-space*** if for distinct points $x, y \in X$ there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

In other words, T_2 means Hausdorff.

Remark 11.2.4. Every T_2-space is a T_1 space.

Definition 11.2.5 (T_3-space, regular space)**.** A space X is T_3 or ***regular*** if X is T_1 and for any closed subset $E \subset X$ and $x \in X \setminus E$ there exist disjoint open sets U and V such that $x \in U$ and $E \subset V$.

Exercise 11.6. Show that a subspace of a regular space is regular. ■

Remark 11.2.6. Since in a T_1-space, single-point sets are closed, T_3 means we can separate single points, so T_3 implies T_2 (or Hausdorff).

Exercise 11.7. Let $\mathscr{B}$ be the collection of subsets of $\mathbb{R}$ of the form (a, b), and $(a, b) \cap \mathbb{Q}$, where $-\infty < a < b < \infty$. Prove the following:

(a) $\mathscr{B}$ is a basis of open sets for a topology $\mathscr{T}$ on $\mathbb{R}$.

(b) $(\mathbb{R}, \mathscr{T})$ is a Hausdorff space.

(c) $\mathbb{R} \setminus \mathbb{Q}$ is $\mathscr{T}$-closed.

(d) Let $f\colon (\mathbb{R}, \mathscr{T}) \to \mathbb{R}$ be a continuous map. If $f = 0$ on $\mathbb{R} \setminus \mathbb{Q}$, then $f = 0$ on $\mathbb{R}$.

(e) $(\mathbb{R}, \mathscr{T})$ is not regular.

This provides an example of a Hausdorff space (T_2) that is not regular (T_2). ■

Definition 11.2.7 (T_4-space, normal space)**.** A space X is T_4 or ***normal*** if X is T_1 and for any disjoint closed subsets E and F there exist disjoint open sets U and V such that $E \subset U$ and $F \subset V$.

Remark 11.2.8. A subspace of a normal space need not be normal.

Remark 11.2.9. Again since a T_4 has to be T_1, single-point sets are closed, so T_4 implies T_3.

In other words,

$$T_4 \implies T_3 \implies T_2 \implies T_1.$$

Theorem 11.2.10. *Every metric space is normal (i.e. T_4).*

Proof. Let (X, d) be a metric space. Obviously X is T_2 (Hausdorff); just take a ball of radius a half the distance between two points.

Let E and F be two disjoint closed sets. For each $x \in E$, there exists a radius $r(x) > 0$ such that $B(x, r(x)) \cap F = \varnothing$ (since F is closed, so $X \setminus F$ is open).

Similarly, for each $y \in F$ there exists a radius $r(y) > 0$ such that $B(y, r(y)) \cap E = \varnothing$.

Then any two $B(x, r(x))$ and $B(y, r(y))$ *might* intersect; but worst case scenario we only need to shrink by a factor of two (inspired by the Hausdorff argument above), so let

$$U = \bigcup_{x \in E} B\Big(x, \frac{r(x)}{2}\Big)$$

and

$$V = \bigcup_{y \in F} B\Big(y, \frac{r(y)}{2}\Big)$$

are open sets. Clearly $E \subset U$ and $F \subset V$.

We claim $U \cap V = \varnothing$. If not, suppose $U \cap V \neq \varnothing$, so there exists $z \in U \cap V$. Then

$$z \in B\Big(x, \frac{r(x)}{2}\Big) \cap B\Big(y, \frac{r(y)}{2}\Big)$$

for some $x \in E$ and $y \in F$. This implies

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r(x)}{2} + \frac{r(y)}{2} \leq \max\{\, r(x), r(y) \,\}.$$

But this means $x \in B(y, r(y))$ or $y \in B(x, r(y))$, which is a contradiction, since $r(x)$ and $r(y)$ were radii specifically chosen to separate x and y. □

11.3 Urysohn's lemma

This asks the question: given two disjoint subsets, can they be separated by a continuous function? The answer is yes, if the space is normal.

To prove this we first need the following:

Lemma 11.3.1. *A topological space X is normal if and only if X is T_1 and given a closed subset $E \subset X$ and an open set $W \supset E$, there exist an open set U such that*

$$E \subset U \subset \overline{U} \subset W.$$

Proof. The forward direction is fairly straightforward. Assume X is normal (so automatically T_1) and suppose $E \subset W$ where E is closed and W is open. Then E and $X \setminus W$ are disjoint closed sets.

Hence, since X is normal, we can separate E and $X \setminus W$ by disjoint open sets U and V such that $E \subset U$ and $(X \setminus W) \subset V$. Thus $\overline{U} \subset (X \setminus V) \subset W$ (since $U \subset X \setminus V$, but $X \setminus V$ is closed).

For the converse, let E and F be disjoint sets. Then $W = X \setminus F \supset E$, and W is open. So by assumption there exists an open set U such that $E \subset U \subset \overline{U} \subset W = X \setminus F$.

Then U and $X \setminus \overline{U}$ are disjoint open sets and $E \subset U$ and $F \subset (X \setminus \overline{U})$. □

Before we state and prove Urysohn's lemma, recall that a ***dyadic rational number*** is a rational number of the form $\frac{m}{2^n}$ where $m, n \in \mathbb{Z}$.

The dyadic rational numbers clearly are dense in $\mathbb{R}$ (e.g. in $[0,1]$, keep taking middle points; they're all dyadic).

Lecture 12 Urysohn's lemma

12.1 Proof of Urysohn's lemma

Theorem 12.1.1 (Urysohn's lemma)**.** *Let E and F be two disjoint closed subsets of a normal space X. Then there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F.*

Remark 12.1.2. The interval $[0,1]$ can be replaced by any closed interval $[a,b]$ so that $f = a$ on E and $f = b$ on F.

Proof. Let $V = X \setminus F$. Since F is closed, V is open, and $E \subset V$. By Lemma 11.3.1, there exist an open set $U_{\frac{1}{2}}$ such that

$$E \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset V.$$

Since E is closed and $U_{\frac{1}{2}}$ is open, we can insert, using the lemma, another pair $U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}}$ between those, and similarly between $\overline{U_{\frac{1}{2}}}$ and V we insert $U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}}$. Hence

$$E \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}} \subset V.$$

Now repeat this process ad nauseam between each closed-open pair, at each finite step adding sets for more and more dyadic rational numbers between 0 and 1. Hence we construct open sets U_r for every dyadic rational $0 < r < 1$ such that

$$\overline{U_r} \subset U_s$$

for $0 < r < s < 1$,

$$E \subset U_r$$

for all $0 < r < 1$, and

$$U_r \subset V$$

for all $0 < r < 1$. Now the idea is to use the boundaries of the U_r as level curves $f = r$ for a function $f\colon X \to [0,1]$. I.e., define $f(x) = 0$ if $x \in U_r$ for all $0 < r < 1$ and $f(x) = \sup\{\, r \mid x \notin U_r \,\}$.

Thus $0 \leq f(x) \leq 1$, $f = 0$ on E, and $f = 1$ on F.

It remains to verify that f is continuous. Let $x \in X$, and let us show that f is continuous at x.

First, suppose $f(x) = 0$. For any $\varepsilon > 0$, take a dyadic rational r such that $0 < r < \varepsilon$. Then $x \in U_r$ (since otherwise $f(x) \geq r > 0$). Hence U_r is an open

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neighbourhood of x. For any $y \in U_r$, $|f(y) - f(x)| = |f(y)| \leq r < \varepsilon$. Hence $f(U_r) \subset B(f(x), \varepsilon)$, so f is continuous at x.

Second, suppose $f(x) = 1$, meaning that $x \notin U_r$ for any $0 < r < 1$ (since otherwise $f(x) \leq r < 1$). Thus $x \notin \overline{U_r}$ (because for $r_1 < r_2$ we have $U_{r_1} \subset \overline{U_{r_1}} \subset U_{r_2} \subset \overline{U_{r_2}}$ by construction, so we can always perturb r slightly). For any $\varepsilon > 0$ there is a dyadic rational s such that $1 - \varepsilon < s < 1$ (so $0 < 1 - s < \varepsilon$).

Since $x \notin \overline{U_s}$, take $W = X \setminus \overline{U_s}$, which is open. Hence W is an open neighbourhood of x.

For any $y \in W$, we have $y \notin U_s$, hence $s \leq f(y) \leq 1$. Hence $0 \leq 1 - f(y) \leq 1 - s < \varepsilon$, so $|f(x) - f(y)| < \varepsilon$. Thus $f(W) \subset B(f(x), \varepsilon)$, so f is continuous at x.

Third, suppose $0 < f(x) < 1$. For any $\varepsilon > 0$, take dyadic rationals r and s such that

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon.$$

Then $x \notin U_t$ for all $r < t < f(x)$. Since $\overline{U_r} \subset U_t$ for $r < t$, this means $x \notin \overline{U_r}$.

On the other hand $x \in U_s$, thus $W = U_s \setminus \overline{U_r}$ is an open neighbourhood of x. For $y \in W$, we have $r \leq f(y) \leq s$, implying that $|f(y) - f(x)| < \varepsilon$. This means $f(W) \subset B(f(x), \varepsilon)$, so f is continuous at x. □

A consequence of this theorem is:

Theorem 12.1.3 (Tietze extension theorem)**.** *Let X be a normal space and let $Y \subset X$ be a closed subset. Let f be a bounded continuous real-valued function on Y. Then there exists a bounded continuous real-valued function h on X such that $h = f$ on Y.*

Proof. We should assume f is not constant on Y, since otherwise we just take h to be the same constant on X.

Step 1: Let $c_0 = \sup_{y \in Y} |f(y)| > 0$ (since f is not constant). Let

$$E_0 = \left\{ y \in Y \;\middle|\; f(y) \leq -\frac{c_0}{3} \right\}$$

and

$$F_0 = \left\{ y \in Y \;\middle|\; f(y) \geq \frac{c_0}{3} \right\}.$$

Then E_0 and F_0 are closed (since they're continuous pullbacks of closed sets) and disjoint subsets of X. By Urysohn's lemma there exists a continuous real-valued function g_0 on X such that $g_0 = -\frac{c_0}{3}$ on E_0 and $g_0 = \frac{c_0}{3}$ on F_0, and

$$-\frac{c_0}{3} \leq g_0(x) \leq \frac{c_0}{3}$$

for all $x \in X$. In particular, $|g_0(x)| \leq \frac{c_0}{3}$ for all $x \in X$, and

$$|f(x) - g_0(x)| \leq \frac{2}{3} c_0$$

for all $x \in Y$ (since f is only defined on Y). In other words we have constructed a continuous function g_0 which approximates f with a maximum error of $\frac{2}{3} c_0$.

Step 2: Construct a sequence $\{\, g_n(x) \,\}_{n=0}^{\infty}$ inductively such that $|g_n(x)| \leq \frac{2^n}{3^{n+1}} c_0$ for all $x \in X$ and

$$\Big| f(x) - \sum_{k=0}^{\infty} g_k(x) \Big| \leq \frac{2^{n+1}}{3^{n+1}} c_0$$

for all $x \in Y$. We do this by considering $f_1(x) = f(x) - g_0(x)$ on Y, let

$$c_1 = \sup_{y \in Y} |f_1(x)| \leq \frac{2}{3} c_0,$$

$$E_1 = \Big\{\, y \in Y \;\Big|\; f)1(y) \leq -\frac{c_1}{3} \,\Big\},$$

and

$$F_1 = \Big\{\, y \in Y \;\Big|\; f_1 \geq \frac{c_1}{3} \,\Big\}.$$

Use step 1 to show there exists a $g_1(x)$ with

$$|g_1(x)| \leq \frac{1}{3} c_1 \leq \frac{2}{3^2} c_0$$

for all $x \in X$ and

$$|\underbrace{f(x) - g_0(x)}_{=f_1(x)} - g_1(x)| \leq \frac{2}{3} c_1 \leq \frac{2^2}{3^2} c_0$$

for all $x \in Y$. Then rinse and repeat with $f_2 = f(x) - g_0(x) - g_1(x)$, and so on.

Step 3: Let

$$h_n(x) = \sum_{k=0}^{n} g_k(x).$$

Then $\{\, h_n(x) \,\}$ converges to a bounded continuous function $h(x)$ on X and $h = f$ on Y.

We show this by showing $\{\, h_n(x) \,\}$ is uniformly Cauchy in X, which means $h_n \to h$ for some continuous function h on X. For $n \geq m$,

$$\begin{aligned}
|h_n(x) - h_m(x)| &= |g_{m+1}(x) + \cdots + g_n(x)| \leq |g_{m+1}(x)| + \cdots + |g_n(x)| \\
&\leq \left(\left(\frac{2}{3}\right)^{m+1} + \cdots + \left(\frac{2}{3}\right)^{n} \right) \frac{c_0}{3} \\
&\leq \left(\frac{2}{3}\right)^{m+1} \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots \right) \frac{c_0}{3} \\
&\leq \left(\frac{2}{3}\right)^{m+1} \frac{1}{1 - \frac{2}{3}} \frac{c_0}{3} = \left(\frac{2}{3}\right)^{m+1} c_0 \to 0
\end{aligned}$$

uniformly (for all $x \in X$) as $m \to \infty$. Hence $\{\, h_n(x) \,\}$ is uniformly Cauchy. Thus $h_n(x) \to h(x)$ is continuous.

Moreover

$$|h(x)| \leq \sum_{n=0}^{\infty} |g_n(x)| \leq \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{c_0}{3} = c_0.$$

Hence h is bounded on X (by the same bound as f on Y).

Finally, for $x \in Y$,

$$|f(x) - h(x)| \le |f(x) - h_n(x)| + |h_n(x) - h(x)| \le \left(\frac{2}{3}\right)^{n+1} c_0 + |h_n(x) - h(x)|.$$

The last term converges to 0 uniformly for all $x \in X$, and the first part also goes to 0 uniformly as n goes to 0. Hence $f(x) = h(x)$ for all $x \in Y$. $\square$

We should always keep in our mind that metric spaces are particular examples of normal spaces.

12.2 The Urysohn metrisation theorem

In general, spaces need not be metric (nor normal), but it is an interesting question to ask whether we can assign a metric fo a space—to metrise it.

There are many results in this direction, we will talk about one of them, namely that a regular space (T_3) with a countable basis (second countable) is metrisable.

Lecture 13 Urysohn's metrisation theorem

13.1 Preliminaries

Lemma 13.1.1. *Every regular space X with a countable basis is normal.*

Proof. Let $\mathscr{B}$ be a countable basis for X. Let E and F be two disjoint closed sets in X. For each $x \in E$, by regularity there exist open sets U and V such that $x \in U$, $F \subset V$, and $U \cap V = \varnothing$.

Since $x \in U \subset (X \setminus V)$, where $X \setminus V$ is closed, $\overline{U} \subset (X \setminus V)$. Thus $\overline{U} \cap F = \varnothing$.

Now since U is open, take $U_x \in \mathscr{B}$ such that $x \in U_x \subset U$. Repeat this for every $x \in E$. Since $\mathscr{B}$ is countable, we have an open cover $\left\{ U_i \right\}_{i=1}^{\infty}$ of E with $\overline{U_i} \cap F = \varnothing$, with $U_i \in \mathscr{B}$.

Similarly F has an open cover $\left\{ V_i \right\}_{i=1}^{\infty}$ with $\overline{V_i} \cap E = \varnothing$ and $V_i \in \mathscr{B}$.

Let

$$U_n' := U_n \setminus \bigcup_{i=1}^{n} \overline{V_i}$$

and

$$V_n' := V_n \setminus \bigcup_{i=1}^{n} \overline{U_i},$$

both open. We claim

$$E \subset U' := \bigcup_{n=1}^{\infty} U_n'$$

and

$$F \subset V' := \bigcup_{n=1}^{\infty} V_n',$$

and importantly $U' \cap V' = \varnothing$.

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For $x \in E$, $x \in U_n$ for some n. Since $\overline{V_i} \cap E = \varnothing$ for all i we have $x \in U'_n$. Hence $x \in U'$, so $E \subset U'$. Similarly $F \subset V'$.

Now suppose $U' \cap V' \neq \varnothing$. In other words, there exists $x \in U' \cap V'$ meaning that $x \in U'_n$ and $x \in V'_m$ for some n and m.

If $n \geq m$, then

$$x \in U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i},$$

so $x \notin \overline{V_m}$, but then $x \notin V'_m$, a contradiction.

Similarly if $n \leq m$, then $x \in V'_m$, meaning $x \notin \overline{U_n}$, so $x \notin U'_n$. □

Lemma 13.1.2. *Let X be a regular space with a countable basis. Then there exist a countable collection of continuous functions $f_n \colon X \to [0,1]$ such that for any $x_0 \in X$ and any neighbourhood U of x_0 there exist an index n such that $f_n(x_0) > 0$ and $f_n = 0$ on $X \setminus U$.*

Proof. Let $\{ B_n \}_{n=1}^{\infty}$ be a countable basis for X. For each pair B_n, B_m with $\overline{B_n} \subset B_m$ we have that $\overline{B_n}$ and $X \setminus B_m$ are disjoint closed sets.

By Lemma 13.1.1, X is normal, and so we can apply Theorem 12.1.1 to construct a continuous function $g_{n,m} \colon X \to [0,1]$ with $g_{n,m}(\overline{B_n}) = 1$ and $g_{n,m}(X \setminus B_m) = 0$.

Reindex $\{ g_{n,m} \}$ to $\{ f_n \}$. Now for $x_0 \in X$ and $x_0 \in U \subset X$ open, we can choose B_n and B_m such that $x_0 \in \overline{B_n} \subset B_m \subset U$ (by regularity).

Then $g_{n,m}(x_0) = 1$ and $g_{n,m}(X \setminus B_m) = 0$. Hence $g(X \setminus U) = 0$ since $B_m \subset U$. □

Now the idea for Urysoh's metrisation theorem is to show that X, using this previous lemma, embeds homeomorphically to the infinite product $\mathbb{R}^{\mathbb{N}} = \prod_{i=1}^{\infty} \mathbb{R}$.

Let

$$\bar{d}(a,b) := \min\{ |a-b|, 1 \}$$

be the ***standard bounded metric*** on $\mathbb{R}$.

Exercise 13.1. Verify that $\bar{d}$ is a metric on $\mathbb{R}$. ■

Let $Y = \mathbb{R}^{\mathbb{N}}$. For $x = (x_i)$ and $y = (y_i)$ in Y, define

$$D(x,y) = \sup_{1 \leq i < \infty} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}.$$

Exercise 13.2. Verify that D is a metric on Y. ■

Proposition 13.1.3. *D is a metric on Y that induces the product topology on $Y = \mathbb{R}^{\mathbb{N}}$. In other words, $\mathbb{R}^{\mathbb{N}}$ (with the product topology) is metrisable.*

Proof. Let U be an open set in the metric topology and let $x = (x_i) \in U$.

We want to show that there exists an open set V in the product topology such that $x \in V \subset U$. (Thus U is open in the product topology.)

Since U is open in the metric topology, choose $\varepsilon > 0$ such that $B_D(x, \varepsilon) \subset U$. Take $N \in \mathbb{N}$ large enough such that $\frac{1}{N} < \varepsilon$. Let

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}.$$

This is open in the product topology (it is a basis element). For $y = (y_i) \in V$, if $i \geq N$ then

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} < \varepsilon.$$

If $i < N$, then

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\varepsilon}{i} < \varepsilon.$$

Thus, taking supremum, $D(x, y) < \varepsilon$, so $y \in B_D(x, \varepsilon)$, meaning that $V \subset B_d(x, \varepsilon) < U$, as desired.

Conversely, let $V = \prod_{i=1}^{\infty} V_i$ be a basis element for the product topology. In other words, $V_i = \mathbb{R}$ for $i > N$ for some $N \in \mathbb{N}$ and V_i is open in $\mathbb{R}$ for $i \leq N$.

For $x = (x_i) \in V$ there exists an open U in the metric topology such that $x \in U \subset V$. To see this, for each $i \leq N$, choose $0 < \varepsilon_i < 1$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset V_i$. Define

$$\varepsilon = \min_{1 \leq i \leq N} \left\{ \frac{\varepsilon_i}{i} \right\}.$$

Let $U = B_D(x, \varepsilon)$. Let $y = (y_i) \in U$. If $i < N$, then $y_i \in V_i = \mathbb{R}$. If $i \leq N$, then

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon < \frac{\varepsilon_i}{i}$$

for all $i \leq N$, so $\bar{d}(x_i, y_i) < \varepsilon_i < 1$. Hence

$$|x_i - y_i| < \varepsilon_i,$$

so $y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) = V_i$. Therefore $U \subset V$. □

We are now equipped to prove:

Theorem 13.1.4 (Urysohn metrisation theorem)**.** *Every regular space X with a countable basis is metrisable.*

Proof. By Lemma 13.1.2, we have a collection of continuous functions $\{ f_n \}$ satisfying the properties in the lemma (chiefly, they separate points in X).

Define $F \colon X \to Y = \mathbb{R}^{\mathbb{N}}$ by

$$F(x) = (f_1(x), f_2(x), \dots).$$

Since f_n is continuous, F is continuous.

For $x \neq y$, there exists f_n such that $f_n(x) > 0 = f_n(y)$ (since the two points can be separated by an open set). Hence $F(x) \neq F(y)$, so F is one-to-one. (I.e., F is an embedding of X into $\mathbb{R}^{\mathbb{N}}$).

We claim that X is homeomorphic to $Z = F(X) \subset Y$. Since Y is metrisable (that's Proposition 13.1.3), so if $F(X)$.

It suffices to show that F^{-1} is continuous (since F is continuous, one-to-one, and onto already). In other words, show that F is an open mapping, i.e., $F(U)$ is open in Z for each open $U \subset X$.

Let $x_0 \in U$ and $F(x_0) = z_0 \in F(U)$. We want to find an open neighbourhood between z_0 and $F(U)$.

Choose an index N such that $f_N(x_0) > 0$ and $f_N(X \setminus U) = 0$ (by construction of the family f_n). Let $V = \pi_N^{-1}((0,\infty))$, where $\pi_N : Y = \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ with $\pi_N((x_i)) = x_N$ is the projection on the Nth coordinate, which is a continuous pullback of an open set, so open in Y.

Let $W = V \cap Z$, open in Z. Then $z_0 \in W$. Now it suffices to show $W \subset F(U)$.

For $z \in W$, $z = F(x)$ for some $x \in X$, $\pi_N(z) = f_N(x) > 0$. Since $F_n(X \setminus U) = 0$, $x \in U$. Thus $z = F(x) \in F(U)$.

Thus finally X is homeomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$, which is metrisable, so X is metrisable as well. □

This idea, contained in the proof, that we can find a family of functions that separate the points, is generally speaking more useful than the Urysohn metrisation theorem itself.

Remark 13.1.5. In fact, we embed X into $[0,1]^{\mathbb{N}}$ since $f_n : X \to [0,1]$.

In fact what we proved in effect is this:

Theorem 13.1.6 (Embedding theorem)**.** *Let X is a T_1-space. Suppose $\{ f_\alpha \}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ such that for each $x \in X$ and neighbourhood U of x, there exists $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha(X \setminus U) = 0$.*

Then $F : X \hookrightarrow \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an embedding of X into $\mathbb{R}^J$.

Lecture 14 Completely regular space and Stone–Čech compactification

14.1 Completely regular space

Definition 14.1.1 (Completely regular)**.** A space X is ***completely regular*** if X is T_1 and for each $x_0 \in X$ and each closed subset $E \subset X$ with $x_0 \notin E$, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(E) = 0$.

Remark 14.1.2. From Lemma 13.1.2 we know that a regular space with countable basis is completely regular.

Remark 14.1.3. From Urysohn metrisation theorem we know that T_4 (normal) implies completely regular.

We also know completely regular implies T_3 (regular), since e.g. if x_0 is a point and E is a closed set, then $x_0 \in f^{-1}((\frac{1}{2}, 1])$ and $E \subset f^{-1}([0, \frac{1}{3}))$ are disjoint open sets.

Hence completely regular is between T_3 and T_4; for this reason (and because T_3 and T_4 got their names before completely regular did), completely regular is referred to as $T_{3.5}$.

Definition 14.1.4 (Compactification)**.** (*i*) A ***compactification*** of a space X is a compact Hausdorff space Y such that $X \subset Y$ and $\overline{X} = Y$.

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(*ii*) Two compactifications Y_1 and Y_2 of X are ***equivalent*** if there exists a homeomorphism $h\colon Y_1 \to Y_2$ such that $h(x) = x$ for all $x \in X$.

Example 14.1.5. As we have previously discussed, $(0, 1)$ has a compactification to $[0, 1]$ and one to S^1 by adding a point at infinity—these are not equivalent. ▲

Proposition 14.1.6. (*i*) *A subspace of a completely regular space is completely regular.*

(*ii*) *Every locally compact Hausdorff space is completely regular.*

Exercise 14.1. Prove Proposition 14.1.6. ■

Lemma 14.1.7. *Let X be a topological space and let Z be a compact Hausdorff space. Let $h\colon X \to Z$ be a continuous embedding (i.e., continuous and one-to-one). Then there exists a corresponding compactification Y of X satisfying the following property:*

There exists an embedding $H\colon Y \to Z$ such that $H = h$ on X. (I.e., we can extend h to an embedding of Y into Z.)

The compactification Y is unique up to equivalence.

Remark 14.1.8. This compactification Y is called the ***compactification induced by*** h.

Proof. The idea is this: since h is an embedding, we identify X with $h(X) = X_0$. Now $Y_0 = \overline{h(X)}$ is closed in Z, but Z is a compact Hausdorff space so Y_0 is also compact Hausdorff. Hence we identify the compactification we want by Y_0, because Y_0 is therefore a compactification of X_0. Now we need to carry this back to X.

Choose a set E disjoint from X that is in one-to-one correspondence with the set Y_0. I.e., $k\colon E \to Y_0$ is one-to-one and onto.

Define $Y = X \sqcup E$, which we want to identify with Y_0. Define $H\colon Y \to Y_0$ by $H(x) = h(x)$ if $x \in X$ and $H(y) = k(y)$ if $y \in E$. Now Y_0 is a topological space, but we don't yet have a topology on Y.

Define a topology on Y by: U is open in Y if and only if $H(U)$ is open in Y_0. Then H is a homeomorphism (it's an open mapping and bijective) and X is a subspace of Y. It is clear that $\overline{X} = Y$. Hence Y is a compactification of X.

Finally, uniqueness: Suppose Y_1 and Y_2 are two compactifications of X with $H_1\colon Y_1 \to Z$ and $H_2\colon Y_2 \to Z$ embeddings, such that $H_i(x) = h(x)$ for all $x \in X$.

Hence $H_i(X) = h(X) = X_0$ for both i, and $H_i(Y_i) = \overline{h(X)} = \overline{X_0} = Y_0$. Hence

$$Y_1 \xrightarrow{H_1} \overline{X_0} \xrightarrow{H_2^{-1}} Y_2$$

is a composition of homeomorphisms, so $H_2^{-1} \circ H_1$ is a homeomorphism of Y_1 and Y_2 and $H_2^{-1} \circ H_1 = \mathrm{Id}_X$ on X. □

Corollary 14.1.9. *A topological space X has a compactification Y if and only if X is completely regular.*

Proof. The forward direction is Proposition 14.1.6.

By the Embedding theorem, we can embed $h\colon X \hookrightarrow [0, 1]^J$. The right-hand side is compact Hausdorff (it's a product of compact Hausdorff spaces, so by

Tychonoff's theorem, infinite case and Proposition 5.2.3 the product is too). By Lemma 14.1.7, X has a compactification induced by h. $\square$

In general, there are many compactifications for X.

Example 14.1.10. Let $X = (0, 1) \subset \mathbb{R}$.

(i) The one-point compactification comes from $h\colon X \to S^1$ by $h(t) = \cos(2\pi t), \sin(2\pi t))$. Then $Y = \overline{X}$.

(ii) Consider $\mathrm{Id}\colon X \to \mathbb{R}$, $\mathrm{Id}(t) = t$. Then $Y = \overline{X} = [0, 1]$. ▲

This raises a question: if Y is a compactification of X, can a continuous real-valued function f on X be extended continuously to Y?

We should assume that f is a bounded function—if it is not, then there is no hope of extending it to Y since Y, being compact, cannot have an unbounded image.

That's not quite enough though:

Example 14.1.11. Consider again $X = (0, 1) \subset \mathbb{R}$.

(i) Let $Y \simeq S^1$ be the one-point compactification of X. A bounded continuous function $f\colon X \to \mathbb{R}$ is extendable to Y if and only if

$$\lim_{x\to 0^+} f(x) \qquad \text{and} \qquad \lim_{x\to 1^-} f(x)$$

both exist and are equal (since in the compactification those endpoints are the same).

(ii) On the other hand, let $Y = [0, 1]$. Then a bounded continuous function $f\colon X \to \mathbb{R}$ is extendable to Y if and only if

$$\lim_{x\to 0^+} f(x) \qquad \text{and} \qquad \lim_{x\to 1^-} f(x)$$

exist. Here they don't need to be equal, since the endpoints are not identified. ▲

The crux is that the definition of $f\colon X \to \mathbb{R}$ is independent of the compactification Y. But given a compactification Y, we can figure out if it's extendable or not.

14.2 Stone–Čech compactification

So, question: can we find a compactification Y so that given *any* bounded continuous function, we can extend it continuously to Y? The answer is yes! This compactification Y is called the ***Stone–Čech compactification*** of X.

The downside is that it is not easy to describe Y precisely.

Theorem 14.2.1 (Stone–Čech compactification, existence)**.** *Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous function $f\colon X \to \mathbb{R}$ extends uniquely to a continuous function on Y.*

Proof. Let $\{ f_\alpha \}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X. For each $\alpha \in J$, define $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X), \subset]\mathbb{R}$ (since f_α is bounded).

Then define $h \colon X \to \prod_{\alpha \in J} I_\alpha$ by $h(x) = (f_\alpha(x))_{\alpha \in J}$.

Since X is completely regular, $\{ f_\alpha \}$ satisfies the condition in the Embedding theorem. Note also that by Tychonoff's theorem, infinite case, $\prod_{\alpha \in J} I_\alpha$ is compact Hausdorff.

Let Y be the compactification of X induced by h. Then we have an embedding

$$H \colon Y \to \prod_{\alpha \in J} I_\alpha$$

with $H = h$ on X.

The construction of Y is independent of X since we built it using the set of *all* bounded continuous real-valued functions on X.

Now given a bounded continuous $f \colon X \to \mathbb{R}$, then $f = f_\beta$ for some $\beta \in J$. Let $\pi_\beta \colon \prod_{\alpha \in J} I_\alpha \to I_\beta$ be the projection map. Then

$$\pi_\beta \circ H \colon Y \xrightarrow{H} \prod_{\alpha \in J} I_\alpha \xrightarrow{\pi_\beta} I_\beta$$

is continuous. For $x \in X$,

$$\pi_\beta \circ H(x) \pi_\beta \circ h(x) = \pi_\beta(f_\alpha(x)_{\alpha \in J}) = f_\beta(x) = f(x),$$

so $\pi_\beta \circ H$ extends $f = f_\beta$ to Y. □

Lecture 15 Analysis

15.1 Stone–Čech compactification

The uniqueness of the extension of f to Y is a consequence of the following lemma:

Lemma 15.1.1. *Let Z be a Hausdorff space. Let $E \subset X$ and let $f \colon E \to Z$ be continuous. Then there is at most one extension of f to a continuous function $F \colon \overline{E} \to Z$.*

Proof. Suppose $F_1, F_2 \colon \overline{E} \to Z$ are two different extensions of f. Thus there exists some $x \in \overline{E} \setminus E$ such that $F_1(x) \neq F_2(x)$. These are in Z, and Z is Hausdorff, so we can separate them: there exist open neighbourhoods U_1 and U_2 of $F_1(x)$ and $F_2(x)$ respectively, and $U_1 \cap U_2 = \varnothing$.

Since F_1 and F_2 are continuous at x, there exist an open neighbourhood V of x such that $F_1(V) \subset U_1$ and $F_2(V) \subset U_2$.

Since $x \in \overline{E}$, this means $V \cap E \neq \varnothing$. Thus there exist $y \in V \cap E$. But $U_1 \ni F_1(y) = f(y) = F_2(y) \in U_2$, contradicting $U_1 \cap U_2 = \varnothing$. □

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Theorem 15.1.2 (Extension property)**.** *Let X be a completely regular space. Let Y be a compactification of X as in Theorem 14.2.1. Let C be a compact Hausdorff space.*

Then for any continuous map $f\colon X \to C$, f extends uniquely to a continuous function $F\colon Y \to C$.

Proof. By Proposition 14.1.6, C is completely regular. By the Embedding theorem, $C \hookrightarrow [0,1]^J$ for some index set J. Hence we may assume $C \subset [0,1]^J$ and $f\colon X \to C$ is given by $f(x) = (f_\alpha(x))_{\alpha\in J}$. Hence each $f_\alpha\colon X \to [0,1]$ is a bounded continuous function. By Theorem 14.2.1, each f_α extends to a continuous function $F_\alpha\colon Y \to \mathbb{R}$.

Define $F\colon Y \to \mathbb{R}^J$ by $F(y) = (F_\alpha(y))_{\alpha\in J}$. This is an extension of f, and $F(y)$ is continuous since each coordinate $F_\alpha(y)$ is continuous. Now since F is continuous,

$$F(Y) = F(\overline{X}) \subset \overline{F(X)} = \overline{f(X)} \subset \overline{C} = C$$

since C is compact in a Hausdorff space, hence closed. □

So if we have a Stone–Čech compactification we can always extend continuous maps with image in a compact Hausdorff space.

Theorem 15.1.3 (Stone–Čech compactification, uniqueness)**.** *Let X be a completely regular space. Suppose Y_1 and Y_2 are two compactifications of X satisfying the extension property Theorem 15.1.2.*

Then Y_1 and Y_2 are equivalent.

Proof. Consider the two inclusion maps $\iota_1\colon X \hookrightarrow Y_1$ and $\iota_2\colon X \hookrightarrow Y_2$.

Since Y_1 has the extension property, ι_2 extends to $F_2\colon Y_1 \to Y_2$, i.e.,

$$\begin{tikzcd} & Y_1 \arrow[d, dashed, "F_2"] \\ X \arrow[ur, hook, "\iota_1"] \arrow[r, hook, "\iota_2=f_2"'] & Y_2 \end{tikzcd}$$

Similarly $\iota_1 = f_1$ extends to $F_1\colon Y_2 \to Y_1$.

Consider

$$F_1 \circ F_2\colon Y_1 \xrightarrow{F_2} Y_2 \xrightarrow{F_1} Y_1.$$

For $x \in X$,

$$F_1 \circ F_2(x) = F_1(\iota_2(x)) = F_1(x) = \iota_1(x) = x,$$

so $F_1 \circ F_2 = \mathrm{Id}$ on X. Hence $F_1 \circ F_2 = \mathrm{Id}$ on Y_1.

Similarly, $F_1 \circ F_2 = \mathrm{Id}$ in Y_2. Hence Y_1 and Y_2 are homeomorphic, because F_1 and F_2 are continuous inverses between the two. □

Definition 15.1.4 (Stone–Čech compactification)**.** The compactification of X in Theorem 14.2.1 is called ***Stone–Čech compactification*** of X, denoted by $\beta(X)$.

It is characterised by the property that any continuous map $f\colon X \to C$, where C is compact Hausdorff, extends uniquely to a continuous map $F\colon \beta(X) \to C$. In a picture,

$$\begin{tikzcd} \beta(X) \arrow[dr, dashed, "F"] & \\ X \arrow[u, hook] \arrow[r, "f"'] & C \end{tikzcd}$$

Exercise 15.1. Let X be a completely regular space and let Y be an arbitrary compactification of X. Let $\beta(X)$ be the Stone–Čech compactification of X. Show that there exists a continuous surjective closed map $f\colon \beta(X) \to Y$ that equals the identity map on X. ■

Remark 15.1.5. This shows that every compactification of X is equivalent to a quotient space of $\beta(X)$.

15.2 Selected topological results in analysis

Definition 15.2.1. Let (X, d) be a metric space.

(*i*) A sequence $\{ x_n \}_{n=1}^{\infty}$ in X is ***Cauchy*** if for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

(*ii*) (X, d) is ***complete*** if every Cauchy sequence in X converges.

Example 15.2.2. For instance, $(\mathbb{R}^n, |\cdot|)$ is a complete metric space. ▲

Exercise 15.2. The space $(\mathbb{R}^{\mathbb{N}}, D)$ is complete, where

$$D(x, y) = \sup_i \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

and

$$\bar{d}(x_i, y_i) = \min\{ |x_i - y_i|, 1 \}.$$ ■

Note that, as we showed in Proposition 13.1.3, the above induces the product topology on $\mathbb{R}^{\mathbb{N}}$, so this space is complete.

Recall that a metric space (X, d) is ***totally bounded*** if for every $\varepsilon > 0$ there exists a finite cover of X by ε-balls.

It is clear that a compact metric space is totally bounded; just cover with ε-balls around every point and use compactness to pick a finite subcover. The converse is not true:

Theorem 15.2.3. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. The forward direction, as mentioned, is trivial. For any $\varepsilon > 0$, $\{ B(x, \varepsilon) \mid x \in X \}$ is an open cover of X. Since X is compact, this cover has a finite subcover, which is the cover we need for totally bounded.

We also need to show it is complete, so let $\{ x_n \}_{n=1}^{\infty}$ be a Cauchy sequence in X. Since X is compact, X is also sequentially compact (since it is a metric space). Hence $\{ x_n \}$ has a convergent subsequence, say $\{ x_{n_k} \}_{k=1}^{\infty}$, converging to say $x \in X$.

Then $x_n \to x$ as well: for any $\varepsilon > 0$, since $x_{n_k} \to x$ as $k \to \infty$, there exists $N_1 \in \mathbb{N}$ such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $n_k \geq N_1$. On the other hand, since $\{ x_n \}$ is Cauchy, there exists $N_2 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq N_2$.

Now let $N = \max\{ N_1, N_2 \}$. Then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $x_n \to x$.

For the converse, assume (X, d) is complete and totally bounded. We claim X is sequentially compact (hence X is compact, since in a metric space they are equivalent).

In other words, we need to show that every sequence has a convergent subsequence. Let $\{ x_n \}_{n=1}^{\infty} \subset X$. This is a standard partition argument; partition the space into parts containing infinitely many terms of the sequence, keep subdividing to identify a convergent subsequence.

So first cover X by finitely many balls of radius 1 (doable since X is totally bounded). Then there exists a ball, say B_1, containing infinitely many x_n (since finitely many balls but infinite sequence). Say $\{ x_n \}_{n \in J_1} \subset B_1$ for some infinite set $J_1 \subset \mathbb{N}$.

Now repeat: consider $\{ x_n \}_{n \in J_1}$ and cover X by finitely many radius $\frac{1}{2}$ balls. At least one ball, say B_2, contains infinitely many $\{ x_n \}_{n \in J_1}$, say $\{ x_n \}_{n \in J_2} \subset B_2$ for some infinite set $J_2 \subset J_1$. Rinse and repeat, so $\{ x_n \}_{n \in J_k}$ with B_k a ball of radius $\frac{1}{2^{k-1}}$ and $J_1 \supset J_2 \supset \dots$.

Take $\{ x_{n_k} \}_{k=1}^{\infty}$ with $n_k \in J_k$. We shall show $\{ x_{n_k} \}$ is Cauchy. Thus $x_{n_k} \to x$ for some $x \in X$ since X is complete.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N-2}} < \varepsilon$. For $k, h > N$, this means $n_k, n_j \in J_N$. Hence $x_{n_k}, x_{n_j} \in B_N$, with radius $\frac{1}{2^{N-1}}$, so

$$d(x_{n_k}, x_{n_h}) < 2\frac{1}{2^{N-1}} = \frac{1}{2^{N-2}} < \varepsilon.$$

Hence $\{ x_{n_k} \}$ is Cauchy. $\square$

15.3 Baire category theorem

Definition 15.3.1 (Baire space). A space X is a ***Baire space*** if for any countable collection of open dense sets $\{ U_n \}_{n=1}^{\infty}$ in X,

$$\bigcap_{n=1}^{\infty} U_n$$

is also dense in X.

The goal of this discussion is to show that a compact Hausdorff space is Baire, and that a complete metric space is also Baire.

Lecture 16 Baire category theorem

16.1 Baire space

We ask what the definition of a Baire space is in terms of closet sets?

Let $E \subset X$ be a subset. Then $X \setminus E$ is dense in X (so $\overline{X \setminus E} = X$) if and only if every point in E is a limit point of $X \setminus E$.

This is true if and only if E contains no open subsets of X other than $\varnothing$, if and only if E has no interior points, so has empty interior.

From this discussion it follows directly that

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Lemma 16.1.1. *A space X is Baire if and only if for any countable collection of closed sets with empty interior $\{ E_n \}_{n=1}^{\infty}$ in X, the union*

$$\bigcup_{n=1}^{\infty} E_n$$

has empty interior.

Theorem 16.1.2 (Baire category theorem)**.** *If X is a compact Hausdorff space or X is a complete metric space, then X is a Baire space.*

Proof. Given a countable collection $\{ E_n \}_{n=1}^{\infty}$ of closed sets withe empty interior in X, we want to show that

$$\bigcup_{n=1}^{\infty} E_n$$

has empty interior. I.e., for any nonempty open set $U \subset X$, there exists $x \in U$ such that $x \notin E_n$ for all n.

Since $\operatorname{int}(E_1) = \varnothing$ and U is open, we must have $U \not\subset E_1$, so there exists $y_1 \in U \setminus E_1$.

Note that our assumption on X implies that X is regular. So there exists an open neighbourhood U_1 of y_1 such that $\overline{U_1} \cap E_1 = \varnothing$, and $\overline{U_1} \subset U$.

Now consider U_1 and E_2. Then by the same argument there exists $y_2 \in U_1 \setminus E_2$ and open neighbourhood U_2 of y_2 such that $\overline{U_2} \cap E_2 = \varnothing$ and $\overline{U_2} \subset U_1$.

Repeating this we obtain a sequence $\{ y_n \}$ and open sets $\{ U_n \}$ such that $y_n \in U_{n-1} \setminus E_n$, U_n is an open neighbourhood of y_n, $\overline{U_n} \cap E_n = \varnothing$, and $\overline{U_n} \subset U_{n-1}$.

Moreover, if X is a metric space, we can also choose U_n such that $\operatorname{diam} U_n < \frac{1}{n}$.

Now we claim

$$\bigcap_{n=1}^{\infty} \overline{U_n} \neq \varnothing.$$

Then there exists some x in the intersection, so $x \notin E_n$ for all n. But $x \in U$ since $x \in U_n$ for all n. Hence, if this claim is true, we are done.

We do this in two cases. First, X is compact Hausdorff. Assume

$$\bigcap_{n=1}^{\infty} \overline{U_n} = \varnothing.$$

Then given $x \in X$, there must exist some n such that $x \notin \overline{U_n}$. Hence $x \in X \setminus \overline{U_n}$, which is open. Hence $\{ X \setminus \overline{U_n} \}_{n=1}^{\infty}$ is an open cover of X. But X is compact, so it has a finite subcover $\{ X \setminus \overline{U_{n_i}} \}_{i=1}^{m}$. I.e.,

$$X = \bigcup_{i=1}^{m} (X \setminus \overline{U_{n_i}}).$$

Taking complement,

$$\varnothing = \bigcap_{i=1}^{m} \overline{U_{n_i}} = \overline{U_{n_m}}$$

since this is a sequence of shrinking sets. But $y_{n_m} \in \overline{U_{n_m}}$, so it is nonempty, which is a contradiction.

Secondly, suppose X is a complete metric space. This is a standard analysis argument. Note that $\{ y_n \}$ is a Cauchy sequence since for $n > N$, $y_n \in U_N$ and $\operatorname{diam}(U_N) < \frac{1}{N}$. Now X is complete, so $y_n \to x$ for some $x \in X$. For any fixed n, $y_k \in U_n$ for all $k \geq n$. Then $y_k \to x$, so $x \in \overline{U_n}$ fora any fixed n. Hence

$$x \in \bigcap_{n=1}^{\infty} \overline{U_n} \neq \varnothing. \qquad \square$$

16.2 Applications of the Baire category theorem

This is clearly a topological result, but it has important applications.

Example 16.2.1. We give a topological proof that the real numbers $\mathbb{R}$ is uncountable.

Suppose $\mathbb{R}$ is countable. In other words, we can write $\mathbb{R} = \{ x_n \}_{n=1}^{\infty}$. Each one-point set is a closed set with empty interior.

But $\mathbb{R}$ is a complete metric space, so by the Baire category theorem it is a Baire space. Hence

$$\bigcup_{n=1}^{\infty} \{ x_n \} = \mathbb{R}$$

has empty interior. But $\mathbb{R}$ obviously has interior, so this is a contradiction. ▲

Another question: Let $f_n \colon [0,1] \to \mathbb{R}$ be continuous such that $f_n(x) \to f(x)$ pointwise.

We know f might not be continuous, since the continuity is not necessarily uniform.

How large is the set $\{ x \in [0,1] \mid f \text{ is continuous at } x \}$?

Lemma 16.2.2. *Let X be a Baire space. Let $Y \subset X$ be an open subspace. Then Y is a Baire space.*

Proof. Let $\{ E_n \}_{n=1}^{\infty}$ be a countable collection of closed sets in Y with empty interior. We need to show

$$\bigcup_{n=1}^{\infty} E_n$$

has empty interior.

Let $\overline{E_n}$ be the closure in X. Then $\overline{E_n} \cap Y = E_n$ since E_n is closed in Y.

Notice that $\overline{E_n}$ has empty interior in X. Otherwise there exists nonempty open $U \subset X$ such that $U \subset \overline{E_n}$, whence $U \cap Y \subset E_n$ and $U \cap Y$ is open in Y, so E_n has nonempty interior.

Since X is Baire,

$$\bigcup_{n=1}^{\infty} \overline{E_n}$$

has empty interior in X.

Now suppose

$$\bigcup_{n=1}^{\infty} E_n \subset Y$$

has nonempty interior in Y. Then there exists some nonempty open W in Y contained in this union. Thus W is also open in X. But then

$$W \subset \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \overline{E_n},$$

which is a contradiction to X being Baire. □

Now we can answer our question.

Theorem 16.2.3. *Let X be a Baire space and let (Y, d) be a metric space. Let $f_n \colon X \to Y$ be continuous for $n = 1, 2, \ldots$. Assume $f_n(x) \to f(x)$ pointwise for some $f \colon X \to Y$.*

Then the set

$$E = \left\{ x \in X \,\middle|\, f \text{ is continuous at } x \right\}$$

is dense in X.

Proof. For any $N \in \mathbb{N}$ and $\varepsilon > 0$, define

$$E_N(\varepsilon) = \left\{ x \in X \,\middle|\, d(f_n(x), f_m(x)) \le \varepsilon \text{ for all } n, m \ge N \right\}.$$

In other words, the set of all x for which the f_n are uniformly close.

For any fixed n and m, let

$$A_{n,m}(\varepsilon) = \left\{ x \in X \,\middle|\, d(f_n(x), f_m(x)) \le \varepsilon \right\}.$$

Note that $A_{n,m}(\varepsilon)$ is closed (it's the pullback of a continuous map).

Thus we have

$$E_n(\varepsilon) = \bigcap_{n,m \ge N} A_{n,m}(\varepsilon)$$

is an arbitrary intersection of closed sets, hence closed. Moreover,

$$E_1(\varepsilon) \subset E_2(\varepsilon) \subset \ldots,$$

and

$$\bigcup_{N=1}^{\infty} E_N(\varepsilon) = X$$

since for any $x \in X$ fixed, $\{ f_n(x) \}$ converges in Y. Hence $\{ f_n(x) \}$ is Cauchy, so the tail is eventually close to each other, and so lies in some $E_N(\varepsilon)$.

Now let

$$U(\varepsilon) := \bigcup_{N=1}^{\infty} \operatorname{int}(E_N(\varepsilon)),$$

which is open in X.

We make two claims. First, $U(\varepsilon)$ is dense in X for any $\varepsilon > 0$.

Second, f is continuous at each point in the set

$$C := \bigcap_{m=1}^{\infty} U\left(\frac{1}{m}\right).$$

Since X is Baire, the first claim implies C is dense in X. Thus the theorem follows by the second claim.

To prove the first claim it suffices to show for any nonempty open V in X, there exists N such that $V \cap \operatorname{int}(E_N(\varepsilon)) \neq \varnothing$.

By Lemma 16.2.2, V is Baire. Second, $V \cap E_N(\varepsilon)$ is closed in V. Now

$$X = \bigcup_{N=1}^{\infty} E_N(\varepsilon),$$

so

$$V = \bigcup_{N=1}^{\infty} (E_N(\varepsilon) \cap V)$$

where $E_N(\varepsilon) \cap V$ are closed in V. Since V is Baire this means $E_N(\varepsilon) \cap V$ has nonempty interior in V for some $N \in \mathbb{N}$. In other words, there exists a nonempty open W in V such that

$$W \subset (E_N(\varepsilon) \cap V).$$

But W is open in V and V is open in X, so W is also open in X. Thus $\varnothing \neq W \subset (\operatorname{int}(E_N(\varepsilon)) \cap V)$, and we are done.

Now to prove the second claim. Let $x \in C$. We want to show f is continuous at x. I.e., for any $\varepsilon > 0$, there exists an open neighbourhood W of x such that $d(f(x), f(y)) < \varepsilon$ for all $y \in W$.

Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\varepsilon}{3}$. Since

$$x \in C = \bigcap_{m=1}^{\infty} U\Big(\frac{1}{m}\Big),$$

we must have $x \in U(\frac{1}{k})$. But

$$U\Big(\frac{1}{k}\Big) = \bigcup_{N=1}^{\infty} \operatorname{int} E_N\Big(\frac{1}{k}\Big).$$

So $x \in \operatorname{int} E_M(\frac{1}{k})$ for some $M \in \mathbb{N}$. Since f_M is continuous at x, there exists an open neighbourhood $W \in \operatorname{int} E_M(\frac{1}{k})$ of x such that $d(f_M(x), f_M(y))\frac{\varepsilon}{3}$ for all $y \in W$. Since $W \subset \operatorname{int} E_M(\frac{1}{k})$, for $n > M$ and $y \in W$,

$$d(f_n(y), f_M(y)) \leq \frac{1}{k} < \frac{\varepsilon}{3}.$$

Now let $n \to \infty$. Then

$$d(f(y), f_M(y)) < \frac{\varepsilon}{3}$$

for all $y \in W$. Thus for $y \in W$,

$$d(f(y), f(x)) \leq d(f(y), f_M(y)) + d(f_M(y), f_M(x)) + d(f_M(x), f(x)) < \varepsilon.$$

Hence f is continuous at x. □

Exercise 16.1. Let X be a complete metric space. Let $\mathcal{F}$ be a subset of the real-valued continuous functions on X. Suppose that for each $x \in X$, the set $\mathcal{F}_x = \{ f(x) \mid f \in \mathcal{F} \}$ is bounded. Show that there exists a non-empty open set U of X such that $\mathcal{F}$ is uniformly bounded on U, i.e. there exists $M \in \mathbb{N}$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and for all $x \in U$. ■

Exercise 16.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $f'\colon \mathbb{R} \to \mathbb{R}$ is continuous on a dense subset of $\mathbb{R}$. Here f' is the usual derivative of f. ■

Lecture 17 Quotient spaces and topological groups

17.1 Quotient spaces

The idea here is that we want to construct a new space from a known space.

Example 17.1.1. If we take a line segment from a point p to q, we can construct a new space by gluing (identifying) p and q.

Given a rectangle we can identify the left-hand side L_1 and the right-hand side L_2 with the same direction, in which case we get a cylinder.

If we now take this cylinder and glue the top circle C_1 to the bottom circle C_2, then we get a torus. ▲

We want to describe this process, this identifying or gluing, in an algebraic way.

Let X be a topological space and let $\sim$ be an equivalence relation on X. This gives us a partition of X into equivalence classes, denoted $X/\sim$, called the ***quotient space*** of X ***modulo*** $\sim$.

Now we want to give a topology on the quotient space, since X originally is a topological space.

Note that we have a natural projection

$$\pi \colon X \to X/\sim$$

by $\pi(x) = [x]$, the equivalence class of x. We would like how new topology to make sure this projection map is continuous.

To this end we define the ***quotient topology*** on $X/\sim$ by $U \subset X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X.

Exercise 17.1. Check that this in fact defined a topology on $X/\sim$. ■

Remark 17.1.2. This is the smallest topology on $X/\sim$ such that π is continuous.

Proposition 17.1.3. *A function $f \colon X/\sim \to Y$ is continuous if and only if $f \circ \pi$ is continuous.*

Exercise 17.2. Prove Proposition 17.1.3. ■

Theorem 17.1.4. *Let X and Y be compact Hausdorff spaces. Let $f \colon X \to Y$ be continuous and surjective. Define an equivalence relation on X by $x_0 \sim x_1$ if and only if $f(x_0) = f(x_1)$.*

Then $X/\sim$ is homeomorphic to Y.

Exercise 17.3. Verify that the relation $\sim$ in Theorem 17.1.4 is an equivalence relation. ■

Proof. In a picture, we want to construct

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

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so that $g\colon X/{\sim} \to Y$ is one-to-one and onto, and is also an open mapping.

To this end we define $g\colon X/{\sim} \to Y$ by $g([x]) \coloneqq f(x)$.

First, this definition depends on the representative x of the equivalence class $[x]$, so we need to check that g is well-defined. Fortunately, this is obvious: if $[x_0] = [x_1]$, i.e., $x_0 \sim x_1$, then by definition $f(x_0) = f(x_1)$.

Second, since $g \circ \pi = f$ and f is continuous, Proposition 17.1.3 implies g is continuous.

Third, g is one-to-one and onto. If $g([x_0]) = g([x_1])$, then $f(x_0) = f(x_1)$, so by definition $x_0 \sim x_1$. In other words, $[x_0] = [x_1]$, so g is one-to-one. Onto is trivial: since f is onto, so is g.

Finally, to show g is a homeomorphism it remains to check that g^{-1} is also continuous. In other words, show that g is an open map (so sends open sets to open sets). For $U \subset X/{\sim}$, we need to show that $g(U)$ is open in Y. This is equivalent to showing $Y \setminus g(U)$ is closed in Y. Since g is onto,

$$Y \setminus g(U) = g(X/{\sim}) \setminus g(U)$$

and since g is one-to-one

$$g(X/{\sim}) \setminus g(U) = g\big((X/{\sim}) \setminus U\big) = f(X \setminus \pi^{-1}(U)).$$

Now $\pi^{-1}(U)$ is open in X since U is open, hence $X \setminus \pi^{-1}(U)$ is closed in X, which is compact Hausdorff. Hence $X \setminus \pi^{-1}(U)$ is compact. Since f is continuous, this means $f(X \setminus \pi^{-1}(U))$ is compact in Y, and since Y is also compact Hausdorff, $f(X \setminus \pi^{-1}(U))$ must be closed in Y.

Thus $g(U)$ has closed complement, so $g(U)$ is open in Y. □

Example 17.1.5. Consider $f\colon [0,1] \to S^1$ by $f(t) = e^{2\pi i t}$, $0 \le t \le 1$.

Note that $f(0) = f(1) = 1$, and these are the only two points with the same image.

Hence the quotient space $[0,1]/{\sim} \cong S^1$, which is the very act of gluing or identifying the endpoints as discussed informally above. ▲

Example 17.1.6. Now consider $X = [0,1] \times [0,1]$. Define $f\colon X \to S^1 \times S^1$ by $f(s,t) = (e^{2\pi i s}, e^{2\pi i t})$, $0 \le s, t \le 1$.

Let us look at when $f(s_0, t_0) = f(s_1, t_1)$. Clearly $f(s_0, 0) = f(s_0, 1)$ since the x-coordinates are the same and the y-coordinates differ by an integer, and the period is 1. Similarly, $f(0, t_0) = f(1, t_0)$.

Hence $X/{\sim}$ is homeomorphic to the torus, and so $S^1 \times S^1$ is in fact a way to describe the torus. ▲

Exercise 17.4. Let $X/{\sim}$ be the quotient space determined by an equivalence relation "$\sim$" on a topological space X. Prove the following:

(a) If X is compact, then $X/{\sim}$ is compact.

(b) If X is connected, then $X/{\sim}$ is connected.

(c) if X is path-connected, then $X/{\sim}$ is path-connected. ■

17.2 Topological group

Definition 17.2.1 (Topological group)**.** A ***topological group*** G is both a T_1 topological space and a group such that the two structures (topological and group) are compatible.

In other words, the group operation $G \times G \to G$ by $(a, b) \mapsto ab$ is continuous (in the sense of the topological structure) and $G \to G$ by $a \mapsto a^{-1}$ is continuous.

Exercise 17.5. In fact, a topological group is *always* a regular space. Prove this! ■

Example 17.2.2. The real numbers $\mathbb{R}$ with the usual topology is a topological group, where the group operation is ordinary addition. ▲

Example 17.2.3. Consider the unit circle $S^1 = \left\{ e^{i\theta} \,\middle|\, 0 \leq \theta \leq 2\pi \right\} \subset \mathbb{R}^2$ with the subspace topology. This is a topological group under multiplication. ▲

Example 17.2.4. Consider $f \colon \mathbb{R} \to S^1$ defined by $f(r) = e^{2\pi i r}$. Then $f(r_0) = f(r_1)$ if and only if $r_0 \equiv r_1 \pmod{\mathbb{Z}}$.

Clearly f is surjective, so the quotient space $\mathbb{R}/{\sim} = \mathbb{R}/\mathbb{Z} \cong S^1$ are homeomorphic.

Now note that S^1 is a topological group under multiplication. On the other hand, $\mathbb{R}$ is an abelian group, so every subgroup is normal. Hence $\mathbb{R}/\mathbb{Z}$ is also a group (under addition). In fact it is a topological group.

Now this means that the homeomorphism above says the *topological* structures of $\mathbb{R}/\mathbb{Z}$ and S^1 are the equivalent, but are the *group* structures? The answer is yes—the $\cong$ above is also a group homomorphism (so in particular an isomorphism). ▲

Proposition 17.2.5. *Let G be a topological group. For any $a \in G$, the left translation $L_a \colon G \to G$ defined by $L_1(g) = ag$ and the right translation $R_a \colon G \to G$ defined by $R_a(g) = ga$ are homeomorphisms.*

Proof. Obvious: L_a and R_a are both one-to-one and onto (their inverses are $L_{a^{-1}}$ and $R_{a^{-1}}$).

It remains to check continuity, which in this case means it suffices to check L_a and R_a are continuous, since they are their own inverses for some a.

But this is obvious:

$$\begin{tikzcd} G \arrow[r, "\text{embed}"] & G \times G \arrow[r, "\text{multiplication}"] & G \\ g \arrow[r, mapsto] \arrow[rr, bend right, "L_a"'] & (a, g) \arrow[r, mapsto] & ag \end{tikzcd}$$

is the composition of two continuous maps (embedding and the group multiplication). Likewise for R_a. □

Corollary 17.2.6. *Let G be a topological group and let V be an open neighbourhood of the identity $e \in G$. Then gV is an open neighbourhood of $g \in G$.*

Proof. This is now obvious: the left translation L_g is a homeomorphism by Proposition 17.2.5, so it sends open sets to open sets. □

Similarly, given an open neighbourhood U of g, we can translate by g^{-1} and get an open neighbourhood of the identity e.

This means that the topology of a topological group is completely determined by neighbourhoods of the identity.

Theorem 17.2.7. *Let G be a topological group. Let K denote the connected component of G containing the identity $e \in G$. Then K is a closed normal subgroup of G.*

Proof. That K is closed is easy: a connected component is always closed in any topological space (this is Exercise 6.4).

Next we need to show that K is a subgroup. In other words, we need to check $ba^{-1} \in K$ for any $a, b \in K$.

For $a \in K$, $Ka^{-1} = R_{a^{-1}}(K)$ is connected (since $R_{a^{-1}}$ is continuous, sending connected sets to connected sets). Moreover $e = aa^{-1} \in Ka^{-1}$. In other words $e \in K$ (since K is the connected component of e). Thus for any $b \in K$, $ba^{-1} \in K$.

Next, K is normal. In other words, for any $g \in G$, $gKg^{-1} \subset K$. But this can be rewritten as

$$gKg^{-1} = L_g(R_{g^{-1}}(K)),$$

which is again connected. It also contains e, since $geg^{-1} = e$. Hence $gKg^{-1} \subset K$. Thus K is indeed a closed normal subgroup of G. □

Theorem 17.2.8. *Let G be a connected topological group. Then for any open neighbourhood V of the identity $e \in G$, the subgroup generated by V is G. In other words, $\langle V \rangle = G$.*

Note that $\langle V \rangle$ is by definition the smallest subgroup of G containing V (it is made up of arbitrary products of elements of V and their inverses).

Lecture 18 Topological groups

Proof. Let $H = \langle V \rangle$. For any $h \in H$, hV is an open neighbourhood of h (since left-translation is a homeomorphism, so sends open set to open set). Moreover $hV \subset H$ since H is a group. Hence every point in H is interior, so H is open in G.

Now we claim $G \setminus U$ is also open, meaning H is closed. But G is connected, so this would mean $H = G$.

For $g \in G \setminus H$, we have $gV \cap H = \varnothing$. Otherwise there exists $x \in gV \cap H$, so $x = gv \in H$ for some $v \in V$, so $h = xv^{-1} \in H$ because $x, v \in H$. Thus $gV \subset G \setminus H$, but gV is an open neighbourhood of g, so $G \setminus H$ is open. □

18.1 Topological groups

Example 18.1.1. Consider $M_n(\mathbb{R})$, the set of all real $n \times n$ matrices. We can view $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ with the usual Euclidean topology.

Consider the subset $\mathrm{GL}_n(\mathbb{R})$ of invertible $n \times n$ matrices with the subspace topology.

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Note that $\mathrm{GL}_n(\mathbb{R})$ with the usual matrix multiplication is a group. In fact, $\mathrm{GL}_n(\mathbb{R})$ is a topological group, in other words the group operations are continuous. First

$$\begin{aligned} m\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) &\longrightarrow M_n(\mathbb{R}) \\ (A, B) &\longmapsto AB \end{aligned}$$

is continuous since each entry of AB is a polynomial in the entries of A and B. Hence it is also continuous when restricted to $\mathrm{GL}_n(\mathbb{R})$.

For $A \mapsto A^{-1}$, the entries aren't quite polynomials in the entries of A, but by Cramer's rule it is a polynomial in the entries of A divided by $\det A$. $A \mapsto \det(A) \mapsto \frac{1}{\det(A)}$ is also continuous, since in $\mathrm{GL}_n(\mathbb{R})$ we avoid determinant 0. ▲

Remark 18.1.2. The map $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is surjective, and by the above discussion also continuous.

A continuous function should send compact sets to compact sets, likewise connected, so $\mathrm{GL}_n(\mathbb{R})$ is *not* compact and *not* connected.

18.2 Orbit spaces

Definition 18.2.1 (Orbit space)**.** A topological group G acts as a group homeomorphism on a set X if each $g \in G$ induces a homeomorphism of X satisfying

(*i*) $g_1 g_2(x) = g_1(g_2(x))$ for all $g_1, g_2 \in G$ and $x \in X$,

(*ii*) $e(x) = x$ for all $x \in X$ where $e \in G$ is the identity of G, and

(*iii*) the map $g\colon X \to X$, $x \mapsto g(x)$ is a homeomorphism.

We define an equivalence relation $\sim$ on X by $x_0 \sim x_1$ if and only if $x_0 = g(x_1)$ for some $g \in G$. In other words, x_0 and x_1 are in the same ***orbit***.

We denote the equivalence classes by $G \setminus X$ (if acting on the left) or X/G (if acting on the right) and endow it with the quotient topology. This space $G \setminus X$ (or X/G) is called the ***orbit space***.

Example 18.2.2. The integers $(\mathbb{Z}, +)$ is a group. Notice how $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\begin{aligned} \mathbb{Z} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (n, r) &\longmapsto n + r. \end{aligned}$$

Hence $r_1 \sim r_2$ if and only if $r_1 - r_2 \in \mathbb{Z}$, describing the orbit space $\mathbb{R}/\mathbb{Z}$. ▲

Definition 18.2.3. If G ***acts transitively*** on X, then X is called a ***homogeneous space*** of G.

The group G acting transitively means for any $x, y \in X$, there exists $g \in G$ such that $g(x) = y$.

Example 18.2.4. Let G be a topological group and let $H < G$ be a subgroup. We know

$$\begin{aligned} H \times G &\longrightarrow G \\ (h, g) &\longmapsto gh^{-1} \end{aligned}$$

is continuous. For each $h \in H$,

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto gh^{-1} \end{aligned}$$

is a homeomorphism (it's right-translation by h^{-1}). So we have the orbit space G/H. ▲

In general, G/H is not a group. Indeed G/H is a group if and only if H is normal in G.

Example 18.2.5. Let $\mathbb{H} = \{ x + iy \in \mathbb{C} \mid x, y \in \mathbb{R},\ y > 0 \}$, the complex upper half-plane. The special linear group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by linear fractional transformation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Hence $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ is an orbit space. ▲

Exercise 18.1. Verify that the linear fractional transformation actually defines a group action. ■

Example 18.2.6. Consider the map $f\colon \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H}$ defined by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g \cdot i = \frac{ai+b}{ci+d}.$$

The map f is surjective since

$$x + iy = \frac{1}{y} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \cdot i.$$

So f induces a homeomorphism $\mathrm{SL}_2(\mathbb{R})/\!\sim\ \cong \mathbb{H}$ where $g_1 \sim g_2$ if and only if $f(g_1) = f(g_2)$. In other words if and only if $g_1 \cdot i = g_2 \cdot i$, so $(g_2^{-1} g_1) \cdot i = i$. In other words $g_2^{-1} g_1 \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i)$, the ***stabiliser*** of i.

Here we mean

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) = \{ g \in \mathrm{SL}_2(\mathbb{R}) \mid g \cdot i = i \} = \mathrm{SO}(2),$$

the ***special orthogonal group***.

So

$$\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2).$$

Hence also

$$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \cong \mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2).$$

So a function on the upper half-plane invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ can be thought of as a function on $\mathrm{SL}_2(\mathbb{R})$ that is left-invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ and right-invariant under the action of $\mathrm{SO}(2)$. ▲

Exercise 18.2. Let G be a topological group and let H be a subgroup of G. Let G/H denote the collection of left cosets with the quotient topology.

(*a*) Show that the projection map $\pi\colon G \to G/H$ is an open map.

(b) Show that H is a closed set in G if and only if G/H is a T_1 space. ■

Exercise 18.3. Let G be a topological group. A neighbourhood V of the identity element e is said to be *symmetric* if $V = V^{-1}$. Here $V^{-1} = \{ v^{-1} \mid v \in V \}$.

(a) Suppose that U is a neighbourhood of e. Show that there exists a symmetric neighbourhood of e such that $V \cdot V \subset U$.

(b) Show that G is a regular topological space. ■

This finishes the discussion of general topology in this course.

Lecture 19 Homotopy theory

19.1 Homotopy of paths

Definition 19.1.1 (Homotopic)**.** Let X and Y be topological spaces. Let $f, g\colon X \to Y$ be continuous. We say f is ***homotopic*** to g, denoted by $f \simeq g$, if there exists a continuous function $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$.

In other words, we can continuously move the image $f(x)$ to the image $g(x)$. I.e., $\gamma_t(x) = F(x,t)\colon X \to Y$ for $0 \le t \le 1$ is a family of continuous functions, continuously deforming from $f(x)$ to $g(x)$.

Now let's consider the special case where f and g are paths in X. Recall if $\gamma\colon [0,1] \to X$ is continuous, $\gamma(0) = x_0$ and $\gamma(1) = x_1$, then γ is called a ***path*** in X from x_0 to x_1.

Definition 19.1.2 (Path homotopic)**.** Two paths γ and γ' in X from x_0 to x_1 are ***path homotopic***, denoted by $\gamma \simeq_p \gamma'$, if there exists a continuous function $F\colon [0,1]\times[0,1] \to X$ such that $F(s,0) = \gamma(s)$ and $F(s,1) = \gamma'(s)$ (so homotopic) and $F(0,t) = x_0$ and $F(1,t) = x_1$ for all $0 \le t \le 1$ (so at every t it is still a path from x_0 to x_1).

Lemma 19.1.3. *The homotopy relation $\simeq$ and the path homotopy relation $\simeq_p$ are equivalence relations on*

$$A = \{ f\colon X \to Y \ \textit{continuous} \}$$

and

$$A(x_0, x_1) = \{ \gamma\colon [0,1] \to X \ \textit{continuous},\ \gamma(0) = x_0,\ \gamma(1) = x_1 \}$$

respectively.

Remark 19.1.4. If γ is a path, we denote its path homotopy equivalence class by $[\gamma]$.

Proof. We will show $\simeq$ is an equivalence relation; path homotopy is very similar.

Reflexivity is obvious: $f \simeq f$ by $F(x,t) = f(x)$ for all $x \in X$ and all $t \in [0,1]$.

For symmetry, suppose $f \simeq g$, say by a continuous $F(x,t)$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Take $G(x,t) = F(x, 1-t)$. Then $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$. Hence $g \simeq f$.

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Finally, transitivity. Suppose $f \simeq g$ and $g \simeq h$, say the first one by $F(x,t)$ and the second by $G(x,t)$. Take

$$H(x,t) = \begin{cases} F(x,2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$. Moreover H is continuous (only $t = \frac{1}{2}$ needs to be checked); it is made up of two continuous functions which agree on a closed set. □

Example 19.1.5. Let γ_1 and γ_2 be two paths from x_0 to x_1 in $\mathbb{R}^2$. Then $\gamma_1 \simeq_p \gamma_2$. For instance, by taking the convex combination of the two paths,

$$F(s,t) = (1-t)\gamma_1(s) + t\gamma_2(s).$$

▲

This argument works in slightly more generality:

Remark 19.1.6. Let γ_1, γ_2 be two paths from x_0 to x_1 in a ***convex*** space X. Then $\gamma_1 \simeq_p \gamma_2$. (Since in a convex space the line segment connecting the two at a fixed time is still in the space because of convexity.)

19.2 Fundamental group

Let γ_0 be a path in X from x_0 to x_1 and let γ_1 be a path in X from x_1 to x_2.

Define $\gamma_0 * \gamma_1$ to be the path from x_0 to x_2 given by

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \gamma_1(2s-1), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

This induces an operation on the path homotopy classes:

$$[\gamma_0] * [\gamma_1] := [\gamma_0 * \gamma_1].$$

Exercise 19.1. Check that $*$ on the path homotopy classes is well-defined (i.e., doesn't depend on the choice of representatives γ_0 and γ_1). ■

Proposition 19.2.1. (*i*) *The operation $*$ is associative. In other words, let γ_0 be a path from x_0 to x_1, γ_1 be a path from x_1 to x_2, and γ_2 a path from x_2 to x_3. Then*

$$([\gamma_0] * [\gamma_1]) * [\gamma_2] = [\gamma_0] * ([\gamma_1] * [\gamma_2]).$$

(*ii*) *The operation $*$ has identities. Given $x \in X$, $e_x \colon [0,1] \to X$, $e_x(s) = x$ be the constant path. Let γ be a path from x_0 to x_1. Then*

$$[e_{x_0}] * [\gamma] = [\gamma] = [\gamma] * [e_{x_1}].$$

(*iii*) *The operation $*$ has inverses. Let γ be a path from x_0 to x_1. Let $\bar{\gamma}(s) := \gamma(1-s)$. Then*

$$[\gamma] * [\bar{\gamma}] = [e_{x_0}] \quad \textit{and} \quad [\bar{\gamma}] * [\gamma] = [e_{x_1}].$$

Exercise 19.2. Prove Proposition 19.2.1. ■

Remark 19.2.2. This means $*$ is a ***groupoid*** operation, but *not* a ***group*** operations, since the left and right identities are not necessarily equal.

On the other hand this means:

Remark 19.2.3. If we consider

$$A(x_0, x_0) = \{\, \gamma\colon [0,1] \to X \text{ continuous},\ \gamma(0) = \gamma(1) = x_0 \,\}$$

then $*$ is a group operation on the path homotopy classes.

Definition 19.2.4. (i) Let X be a space and let $x_0 \in X$. A path in X that begins and ends at x_0 is called a ***loop*** at x_0.

(ii) The set of path homotopy classes of loops based at x_0 with the operation $*$ is called the ***fundamental group*** of X relative to the base point x_0, denoted by $\pi_1(X, x_0)$.

Example 19.2.5. For any $x_0 \in \mathbb{R}^2$, $\pi_1(\mathbb{R}^2, x_0) = \{\, e \,\} = 0$, the trivial group, since all paths in $\mathbb{R}^2$ are path homotopic by Example 19.1.5. ▲

In general, if X is convex, then $\pi_1(X, x_0) = 0$ for all $x_0 \in X$. In particular, $\pi_1(\mathbb{R}^n, x_0) = 0$.

19.3 Changing base point

Let $x_0, x_1 \in X$ and let α be a path from x_0 to x_1. Then α induces a group homomorphism

$$\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by

$$\hat{\alpha}([\gamma]) = [\bar{\alpha}] * [\gamma] * [\alpha] = [\bar{\alpha} * \gamma * \alpha].$$

(Recall $\bar{\alpha}(s) = \alpha(1 - s)$ is α in reverse.)

Exercise 19.3. Verify that $\bar{\alpha}$ is a group homomorphism. ■

Theorem 19.3.1. *Even better: $\hat{\alpha}$ is a group isomorphism.*

Proof. Let $\beta(s) = \bar{\alpha}(s)$. This is a path from x_1 to x_0. Then $\hat{\beta}\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is a group homomorphism, and $\hat{\alpha}$ and $\hat{\beta}$ are each others inverses. □

Lecture 20 Covering spaces

20.1 Simply connected space

Corollary 20.1.1. *If X is path connected, then for any $x_0, x_1 \in X$, we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Remark 20.1.2. This isomorphism depends on the path α from x_0 to x_1.
Two different paths may induce different isomorphisms.

Definition 20.1.3 (Simply connected)**.** A space X is ***simply connected*** if X is path connected and $\pi_1(X, x_0) = 0$.

Note that since the space is path connected, the fundamental group $\pi_1(X, x_0)$ does not depend on the choice of $x_0 \in X$ in the first place.

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20.2 Induced homomorphism

Let $h\colon X \to Y$ be a continuous map such that $h(x_0) = y_0$. Write $h\colon (X, x_0) \to (Y, y_0)$. Then h induces a homomorphism $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by

$$h_*([\gamma]) = [h \circ \gamma].$$

Exercise 20.1. Verify that h_* is a group homomorphism. ■

Theorem 20.2.1. *Let $h\colon (X, x_0) \to (Y, y_0)$ and $k\colon (Y, y_0) \to (Z, z_0)$ be continuous. Then $(k \circ h)_* = k_* \circ h_*$.*

Exercise 20.2. Prove Theorem 20.2.1. ■

Corollary 20.2.2. *If $h\colon (X, x_0) \to (Y, y_0)$ is a homeomorphism, then*

$$h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Hence the fundamental group π_1 is a topological invariant.

20.3 Covering spaces

Definition 20.3.1 (Covering space)**.** (*i*) Let $p\colon E \to X$ be a continuous surjective map. An open set $U \subset X$ is ***evenly covered*** by p if $p^{-1}(U)$ is a union of disjoint open subsets $V_\alpha \subset E$ such that

$$p\big|_{V_\alpha}\colon V_\alpha \to U$$

is a homeomorphism for all α.

(*ii*) Let $p\colon E \to X$ be a continuous surjective map. If each $x \in X$ has a neighbourhood U that is evenly covered by p, then p is called a ***covering map***, and E is called a ***covering space*** of X.

Example 20.3.2. Consider $p\colon \mathbb{R} \to S^1$ defined by $p(t) = e^{2\pi i t}$. This p is a covering map. ▲

Definition 20.3.3 (Fibre)**.** Let $p\colon E \to X$ be a covering map. Let $x \in X$. Then $p^{-1}(x)$ is called the ***fibre*** over x.

Remark 20.3.4. The fibre $p^{-1}(x)$ has the discrete topology, and for each $x \in X$ there is an open neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(x) \times U$.

20.4 Lifting and universal covering spaces

Definition 20.4.1. Let $p\colon E \to X$ be a covering map. Let $f\colon Y \to X$ be a continuous map. A continuous map $g\colon Y \to E$ is called a ***lift*** of f if $p \circ g = f$. In other words, a lift is a map making the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

commute.

Lemma 20.4.2 (Uniqueness of lifts)**.** *Let $p\colon E \to X$ be a covering map and Y be a connected space. Let $f\colon Y \to X$ be a continuous map and let $g, h\colon Y \to E$ be two lifts of f.*

Suppose $g(y_0) = h(y_0)$ for some $y_0 \in Y$. Then $g(y) = h(y)$ for all $y \in Y$.

Proof. Let $A = \{\, y \in Y \mid g(y) = h(y) \,\}$ and $B = \{\, y \in Y \mid g(y) \neq h(y) \,\}$. Obviously $y_0 \in A$, so $A \neq \varnothing$. Moreover $Y = A \sqcup B$.

We claim A and B are open. Thus $A = Y$ since Y is connected and $A \neq \varnothing$. In other words, $g = h$.

Let $y \in Y$ and U be an open neighbourhood of $f(y)$ that is evenly covered by p. In other words

$$p^{-1}(U) = \bigsqcup_{\alpha} V_\alpha$$

and $p\big|_{V_\alpha}\colon V_\alpha \to U$ is a homeomorphism.

We have two cases: if $y \in A$, i.e., $g(y) = h(y) \in p^{-1}(U)$, then $g(y) = h(y) \in V_\alpha$ for some α. Now g and h are continuous, so there exists an open neighbourhood W of y such that $g(W) \subset V_\alpha$ and $h(W) \subset V_\alpha$.

For any $z \in W$, $p \circ g(z) = f(z) = p \circ h(z)$. Since $p\big|_{V_\alpha}$ is a homeomorphism, it is in particular one-to-one, so $g(z) = h(z)$. Thus $y \in W \subset A$, so A is open.

Second case, suppose $y \in B$, so $g(y) \neq h(y)$. Since $p \circ g(y) = f(y) = p \circ h(y)$, the points $g(y)$ and $h(y)$ cannot be in the same V_α. Hence $g(y) \in V_\alpha$ and $h(y) \in V_\beta$ for some $\alpha \neq \beta$. Thus $V_\alpha \cap V_\beta = \varnothing$.

Since g and h are continuous, there exists an open neighbourhood W of y such that $g(W) \subset V_\alpha$ and $h(W) \subset V_\beta$. So for $z \in W$, $g(z) \neq h(z)$. Hence $W \subset B$, so B is open. □

This means that a map from a connected space has unique lifts, if a lift exists at all. But do they exist?

20.5 Existence of path lifting

Theorem 20.5.1 (Path lifting theorem)**.** *Let $p\colon E \to X$ be a covering map. Let $\gamma\colon [0,1] \to X$ be a path and let $e_0 \in E$ such that $p(e_0) = \gamma(0)$. Then there exists a unique path $\alpha\colon [0,1] \to E$ such that $\alpha(0) = e_0$ and $p \circ \alpha = \gamma$ (so α is a lift of γ).*

Proof. The uniqueness follows from the lemma since we fixed the starting point e_0.

On to existence. The idea is simple: pull back the path γ on the neighbourhoods on which we have one-to-one and onto, due to the covering map.

That is, for each $x \in X$, choose an open neighbourhood U_x of x that is evenly covered by p. Then $\{\, \gamma^{-1}(U_x) \,\}_{x \in X}$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, by Lebesgue number lemma there exist $0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$ and $U_1, U_2, \ldots, U_m$ such that $[s_{j-1}, s_j] \subset \gamma^{-1}(U_j)$.

Since $p(e_0) = \gamma(0) \in U_1$, there is an open neighbourhood V_1 of e_0 such that $p\big|_{V_1}\colon V_1 \to U$ is a homeomorphism. Define the lift α of γ on $[0, s_1]$ by $\alpha(t) = (p\big|_{V_1})^{-1}(\gamma(t))$ for $0 \leq t \leq s_1$.

Since $\gamma([s_1, s_2]) \subset U_2$, we have $\gamma(s_1) \in U_2$. Set $e_1 = \alpha(s_1) \in E$. There exists an open neighbourhood V_2 of e_1 such that $p\big|_{V_2}\colon V_2 \to U_2$ is a homeomorphism. Define α on $[s_1, s_2]$ by $\alpha(t) = (p\big|_{V_2})^{-1}(\gamma(t))$ for $s_1 \leq t \leq s_2$.

Continuing this process (in finitely many steps since $[0,1]$ is compact), we have a lift α of γ. $\square$

Lecture 21 Universal covering space

21.1 Lifting

Theorem 21.1.1 (Lifting of path homotopy). *Let $p\colon E \to X$ be a covering map. Let $F\colon [0,1]\times[0,1] \to X$ be a continuous map. Let $e_0 \in E$ satisfy $p(e_0) = F(0,0) = x_0$. Then there exists a unique lift $G\colon [0,1]\times[0,1] \to E$ of F such that $G(0,0) = e_0$.*

Moreover, if F is a path homotopy, then G is a path homotopy.

Proof. The uniqueness follows from Lemma 20.4.2.

Now existence. The idea is similar to path lifting in Theorem 20.5.1. Consider the path $s \mapsto F(s,0)$. By the path lifting theorem, there exists a unique path $\alpha(s)$ in E such that $p\circ\alpha(s) = F(s,0)$ and $\alpha(0) = e_0$.

Similarly, consider the path $t \mapsto F(0,t)$. Again there exists a unique lift $\beta(t)$ in E such that $p \circ \beta(t) = F(0,t)$ and $\beta(0) = e_0$.

Use the Lebesgue number lemma, so there exist $0 < s_0 < s_1 < s_2 < \cdots < s_m = 1$ and $0 < t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that

$$F([s_{i-1}, s_i] \times [t_{j-1}, t_j]) \subset U_{ij}$$

in X that is evenly covered by p.

First consider $F([0,s_1]\times[0,t_1]) \subset U_{00}$. Let $V_{00} \subset E$ such that $p\big|_{V_{00}}\colon V_{00} \to U_{00}$ is a homeomorphism such that $\alpha([0,s_1]) \subset V_{00}$ and $\beta([0,t_1]) \subset V_{00}$.

Define the lift G on F on $[0,s_1]\times[0,t_1]$ by the pullback

$$G(s,t) = (p\big|_{V_{00}})^{-1}(F(s,t)).$$

Next consider

$$F([s_1,s_2]\times[0,t_1]) \subset U_{10}.$$

Let $V_{10} \subset E$ such that

$$p\big|_{V_{10}}\colon V_{10} \to U_{10}$$

is a homeomorphism and $\alpha([s_1,s_2]) \subset V_{10}$ and $\beta([0,t_1]) \subset V_{10}$. Define G on $[s_1,s_2]\times[0,t_1]$ by the pullback

$$G(s,t) = (p\big|_{V_{10}})^{-1}(F(s,t))$$

Continue this process for each block $[s_{i-1},s_i]\times[t_{i-1},t_i]$ to fill out $[0,1]\times[0,1]$. We get a lift G of F defined on $[0,1]\times[0,1]$ such that $p\circ G = F$ and $G(0,0) = e_0$.

Now assume F is a path homotopy. So say $F(0,t) = x_0$ and $F(1,t) = x_1$ for all $t \in [0,1]$.

Since $p(G(0,t)) = F(0,t) = x_0$ for all t, $G(\{0\}\times[0,1]) \subset p^{-1}(x_0)$. On the one hand, $p^{-1}(x_0)$ is discrete, but $\{0\}\times[0,1]$ is connected and G is continuous, so the left-hand side is continuous. Hence $G(\{0\}\times[0,1])$ is a single point set, so $G(0,t) = G(0,0) = e_0$ for all $t \in [0,1]$.

Similarly, $G(1,t) = G(1,0)$ for all t, so G is a path homotopy. $\square$

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Remark 21.1.2. Let $p\colon (E, e_0) \to (X, x_0)$ be a covering map.

First, let $\gamma\colon [0,1] \to X$ be a loop based at x_0. (I.e., $\gamma(0) = \gamma(1) = x_0$.)

By the Path lifting theorem, there exists a unique $\alpha\colon [0,1] \to E$ such that $p \circ \alpha = \gamma$ and $\alpha(0) = e_0$.

Now $p \circ \alpha(1) = \gamma(1) = x_0$, so $\alpha(1) \in p^{-1}(x_0)$. However $\alpha(1)$ need *not* be e_0.

For example, $p(t) = e^{2\pi i t}$ from the previously example applied to $\gamma(t) = e^{2\pi i t}$ has $\alpha(0) = 0$ and $\alpha(1) = 1$.

Second, suppose two loops $\gamma_1, \gamma_2\colon [0,1] \to X$ based at x_0 are homotopic Then their lifts $\alpha_1, \alpha_2\colon [0,1] \to E$ are path homotopic. Thus $\alpha_1(1) = \alpha_2(1)$ (but we don't know if they're loops).

So we can define a function $\Phi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ by $[\gamma] \mapsto \alpha(1)$ where α is a lift of γ. Note by the previous paragraph Φ is well-defined.

Theorem 21.1.3. *Let $p\colon (E, e_0) \to (X, x_0)$ be a covering map. Suppose E is simply connected. Then $\Phi\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is one-to-one and onto.*

Proof. Let's start with onto. For $y \in p^{-1}(x_0)$, let α be a path in E from e_0 to y (which exists since E is simply connected, hence path connected). Let $\gamma = p \circ \alpha$, then γ is a loop in X based at x_0, and α is a lift of γ such that $\alpha(0) = e_0$. Thus $\Phi([\gamma]) = \alpha(1) = y$, so Φ is onto $p^{-1}(x_0)$.

Next one-to-one. Suppose γ_0 and γ_1 are two loops in X based at x_0 such that $\Phi([\gamma_0]) = \Phi([\gamma_1])$. We want to show $[\gamma_0] = [\gamma_1]$, in other words $\gamma_0 \simeq \gamma_1$.

Let α_0 and α_1 be lifts of γ_0 and γ_1 respectively, such that $\alpha_0(0) = \alpha_1(0) = e_0$.

Thus $\alpha_0 * \bar{\alpha}_1$ is a loop in E based at e_0.

Since E is simply connected, there exists a homotopy $F\colon [0,1] \times [0,1] \to E$ such that $F(0,t) = \alpha_0 * \bar{\alpha}_1(t)$ and $F(1,t) = e_0$.

Projecting this homotopy, $p \circ F\colon [0,1] \times [0,1] \to X$ is a homotopy of the loop $\gamma_0 * \bar{\gamma}_1$ and the constant map x_0. Hence

$$[\gamma_0 * \bar{\gamma}_1] = [x_0]$$

and $[x_0]$ is the identity in $\pi_1(X, x_0)$. But $[\bar{\gamma}_1] = [\gamma_1]^{-1}$, so $[\gamma_0] = [\gamma_1]$. Hence Φ is one-to-one. $\square$

Definition 21.1.4 (Universal covering space)**.** Suppose $p\colon (E, e_0) \to (X, x_0)$ is a covering map. If E is simply connected, then E is called a ***universal covering space*** of X.

Remark 21.1.5. Not every space has a universal covering space.

Consequently, a question: When does a universal covering space exist? The answer is fairly technical and we won't go into it here beyond stating the result:

Definition 21.1.6 (Semilocally simply connected)**.** A space X is called ***semilocally simply connected*** if for each $x_0 \in X$ there exists a neighbourhood U of x such that the homomorphism $i_*\colon \pi_1(U, x_0) \to \pi_1(X, x_0)$ induced by the inclusion map $i\colon U \hookrightarrow X$ is trivial.

Theorem 21.1.7. *A space X has a universal covering space if and only if X is path connected, locally path connected, and semilocally simply connected.*

The point is this: if there is a universal covering space of X, then the fundamental group $\pi_1(X, x_0)$ is in one-to-one correspondence with the fibre $p^{-1}(x_0)$.

We'll use this to show $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Exercise 21.1. Let $p\colon (E, e_0) \to (X, x_0)$ be a covering map and let E be path-connected. Prove the following:

(a) $p_*\colon \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective.

(b) There is a bijection $\phi\colon H \setminus \pi_1(X, x_0) \to p^{-1}(x_0)$ where $H = p_*(\pi_1(E, e_0))$ and $H \setminus \pi_1(X, x_0)$ is the collection of right cosets of H in $\pi_1(X, x_0)$. ■

Lecture 22 Homotopy type

22.1 Fundamental group of the unit circle

Example 22.1.1. We will show $\pi_1(S^1, 1) \cong \mathbb{Z}$ as groups.

Consider the covering map $p\colon (\mathbb{R}, 0) \to (S^1, 1)$ given by $p(t) = e^{2\pi i t}$.

Note that since $\mathbb{R}$ is simply connected (since $\mathbb{R}$ is convex), p is a universal covering map. Secondly, $p^{-1}(1) = \mathbb{Z}$.

So $\Phi\colon \pi_1(S^1, 1) \to \pi^{-1}(1) = \mathbb{Z}$ is one-to-one and onto. We also want to show, then, that Φ is a group isomorphism.

Let γ be a loop in S^1 based at 1. Let α be a lift of γ in $\mathbb{R}$ with $\alpha(0) = 0$. Then $p \circ \alpha(1) = \gamma(1) = 1$, so in other words $e^{2\pi i \alpha(1)} = 1$, so $\alpha(1) \in \mathbb{Z}$.

Define the index of γ by

$$\operatorname{ind}(\gamma) = \alpha(1) \in \mathbb{Z},$$

so the index counts how many times γ makes full revolutions (with sign). Let γ_1 and γ_2 be two loops in S^1 based at 1. Since Φ is one-to-one, $[\gamma_1] = [\gamma_2]$ if and only if $\operatorname{ind}(\gamma_1) = \operatorname{ind}(\gamma_2)$.

Now we claim $\Phi\colon \pi_1(S^1, 1) \to \mathbb{Z} = p^{-1}(1)$ given by $[\gamma] = \operatorname{ind}(\gamma)$ is a group isomorphism.

In other words, we want to show

$$\operatorname{ind}(\gamma_1 * \gamma_2) = \operatorname{ind}(\gamma_1) + \operatorname{ind}(\gamma_2).$$

Let α_1 and α_2 be lifts of γ_1 and γ_2 respectively, such that $\alpha_1(0) = \alpha_2(0) = 0$.

Thus $p \circ \alpha_1 = \gamma_1$ and $p \circ \alpha_2 = \gamma_2$, or on other words $\gamma_1(t) = e^{2\pi i \alpha_1(t)}$ and $\gamma_2(t) = e^{2\pi i \alpha_2(t)}$.

We want to compute $\operatorname{ind}(\gamma_1 * \gamma_2)$, so first we need a lift of $\gamma_1 * \gamma_2$. Define $\beta(t)$ as

$$\beta(t) = \begin{cases} \alpha_1(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \alpha_1(1) + \alpha_2(2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\beta(t)$ is continuous, and indeed β is a lift of $\gamma_1 * \gamma_2$ and $\beta(0) = \alpha_1(0) = 0$.

Then

$$\operatorname{ind}(\gamma_1 * \gamma_2) = \beta(1) = \alpha_1(1) + \alpha_2(1) = \operatorname{ind}(\gamma_1) + \operatorname{ind}(\gamma_2). \qquad \blacktriangle$$

Date: November 10th, 2020.

22.2 Homotopy type

Recall that $f\colon (X, x_0) \to (Y, y_0)$, continuous, induces a group homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\gamma] \mapsto [f \circ \gamma]$.

Recall also that if X is path connected, then $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a (noncanonical) isomorphism.

Theorem 22.2.1. *Let $F\colon X \times [0, 1] \to Y$ be a homotopy of two maps $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (X, x_0) \to (Y, y_1)$. (I.e., $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.)*

Let $\alpha(t) = F(x_0, t)$, $0 \le t \le 1$, a path from y_0 to y_1.

Then $g_* = \hat{\alpha} \circ f_*$. In other words, the diagram

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, y_0) \arrow[d, dashed, "\hat{\alpha}\ \textit{isomorphism}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

commutes. In particular, f_* is an isomorphism if and only if g_* is an isomorphism.

Proof. Let $\gamma\colon [0, 1] \to X$ be a loop based at x_0. We need to show $\hat{\alpha}(f_*([\gamma])) = g_*([\gamma])$. In other words, show

$$[\bar{\alpha}] * [f \circ \gamma] * [\alpha] = [g \circ \gamma].$$

Note that f can be moved to g by the homotopy F. Consider therefore the map $G\colon [0, 1] \times [0, 1] \to Y$ defined by

$$G(s, t) = F(\gamma(s), t).$$

Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the paths along the edges of $[0, 1] \times [0, 1]$: β_1 to the right along the bottom, β_2 going up along the left-hand side, β_4 going up along the right-hand side, and β_3 going to the right along the top.

So $G \circ \beta_1 = F(\gamma(s), 0) = f \circ \gamma$, $G \circ \beta_2 = F(\gamma(0), t) = F(x_0, t) = \alpha$. We have $G \circ \beta_3 = F(\gamma(s), 1) = g \circ \gamma$, and finally $G \circ \beta_4 = F(\gamma(1), t) = F(x_0, t) = \alpha$.

Note that $[0, 1] \times [0, 1]$ is convex. Hence any path in the square is homotopic to any other path with the same start and end points. Hence $\beta_3 \simeq \bar{\beta}_2 * \beta_1 * \beta_4$. Compose with G, so

$$G \circ \beta_3 \simeq G \circ (\bar{\beta}_2 * \beta_1 * \beta_4),$$

so in other words

$$[g \circ \gamma] = [\bar{\alpha} * (f \circ \gamma) * \alpha]. \qquad \square$$

Definition 22.2.2 (Homotopy type)**.** Let X and Y be topological spaces.

(i) A map $f\colon X \to Y$ is a ***homotopy equivalence*** if there exists a map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$.

The map g is called a ***homotopy inverse*** of f.

(ii) The spaces X and Y are ***homotopy equivalent*** (or have the same ***homotopy type***) if there is a homotopy equivalence between X and Y.

Theorem 22.2.3. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Let g be a homotopy inverse of f, and $x_1 = g(y_0)$, i.e. $g\colon (Y, y_0) \to (X, x_1)$.

So

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] \arrow[rr, bend right, "g\circ f\simeq \mathrm{Id}_X"'] & (Y, y_0) \arrow[r, "g"] & (X, x_1) \end{tikzcd}$$

Then

$$(g \circ f)_* = g_* \circ f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0) \to \pi_1(X, x_1),$$

but $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x_0)}$.

By Theorem 22.2.1, $g_* \circ f_* = \mathrm{Id}_X$ so f_* is one-to-one and g_* is onto. Similarly, consider $f \circ g$, saying f_* is onto and g_* is one-to-one. □

Remark 22.2.4. If X and Y are homeomorphic, then X and Y have the same homotopy type.

The converse is not true: for instance a single point $\{ x_0 \}$ is homotopy equivalent to $\mathbb{R}$, but they are certainly not homeomorphic.

Definition 22.2.5 (Contractible)**.** A space X is ***contractible*** if X is homotopy equivalent to a single-point space $Y = \{ y_0 \}$.

Corollary 22.2.6. *A contractible space is simply connected.*

Exercise 22.1. Prove that if X is contractible and Y is path connected, then any two maps from X to Y are homotopic. ■

Exercise 22.2. Let X be a path-connected space and let $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is abelian if and only if for any paths α, β from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$. ■

Exercise 22.3. Let X be a topological space and let $x_0 \in X$. Suppose that there is a continuous map $F\colon X \times [0, 1] \to X$ such that

$$\begin{aligned} F(x, 0) &= x_0, \quad x \in X, \\ F(x, 1) &= x, \quad x \in X, \\ F(x_0, t) &= x_0, \quad 0 \le t \le 1. \end{aligned}$$

(*a*) Show that X is path connected.

(*b*) Show that X is simply connected. ■

Exercise 22.4. A subset A of $\mathbb{R}^n$ is called *star convex* with respect to $a_0 \in A$ if all the line segments joining a_0 to any other points of A lie in A.

(*a*) Find a star convex set that is not convex.

(*b*) Show that a star convex set is simply connected. ■

Exercise 22.5. Let X be a simply connected topological space. Let $x_0, x_1 \in X$. Show that any two paths from x_0 to x_1 are homotopic. ■

Lecture 23 Fundamental group calculations

23.1 Fundamental group of the punctured plane

Theorem 23.1.1. *Let $x_0 \in S^1$. Let $0 = (0,0) \in \mathbb{R}^2$. The inclusion map $\iota\colon (S^1, x_0) \to (\mathbb{R}^2 \setminus \{0\}, x_0)$ induces an isomorphism $\iota_*\colon \pi_1(S^1, x_0) \to \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$.*

Hence $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) = \mathbb{Z}$.

Remark 23.1.2. Since $\mathbb{R}^2 \setminus \{0\}$ is path-connected, we can replace x_0 by any base point $x \neq 0$ above.

Proof. It suffices to show ι is an homotopy equivalence. Define the continuous map $r\colon \mathbb{R}^2 \setminus \{0\} \to S^1$ by $r(x) = \frac{x}{\|x\|}$. Then $r \circ \iota = \mathrm{Id}_{S^1}$ quite obviously. We need to also show

$$\iota \circ r\colon \mathbb{R}^2 \setminus \{0\} \xrightarrow{r} S^1 \xrightarrow{\iota} \mathbb{R}^2 \setminus \{0\}$$

is homotopic to $\mathrm{Id}_{\mathbb{R}^2 \setminus \{0\}}$. Note

$$\iota \circ r(x) = \iota\Big(\frac{x}{\|x\|}\Big) = \frac{x}{\|x\|}.$$

Define $F\colon \mathbb{R}^2 \setminus \{0\} \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$ by

$$F(x,t) = t\frac{x}{\|x\|} + (1-t)x.$$

Then $F(x,0) = x = \mathrm{Id}_{\mathbb{R}^2 \setminus \{0\}}(x)$ and $F(x,1) = \frac{x}{\|x\|} = \iota \circ r(x)$, so $\iota \circ r \simeq \mathrm{Id}_{\mathbb{R}^2 \setminus \{0\}}$. □

Using the same argument, we can prove

Theorem 23.1.3. *Let $x_0 \in S^{n-1}$. The inclusion map $\iota\colon (S^{n-1}, x_0) \to (\mathbb{R}^n \setminus \{0\}, x_0)$ induces an isomorphism $\iota_*\colon \pi_1(S^{n-1}, x_0) \to \pi_1(\mathbb{R}^n \setminus \{0\}, x_0)$.*

This raises the question: what is (S^{n-1}, x_0) for $n \geq 3$? The answer is 0. We will show this later.

23.2 Fundamental group of S^n

Theorem 23.2.1 (van Kampen theorem (special case))**.** *Let $X = U \cup V$ where U and V are open sets in X. Suppose U and V are both simply connected and $U \cap V$ is path connected. Let $x_0 \in U \cap V$. Then $\pi_1(X, x_0) = 0$. (So X is simply connected.)*

Sketch of proof. Let $f\colon [0,1] \to X$ be a loop based at x_0. We need to show f is path-homotopic to the constant map x_0.

Step 1: By the Lebesgue number lemma there exists a subdivision of $[0,1]$, say

$$0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$$

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such that $f([a_i, a_{i+1}]) \subset U$ or V (since $U \cup V$ is an open cover of the compact set $f([0,1])$). We can choose a_i such that $f(a_i) \in U \cup V$ for all i. (Namely, remove a_i and extent the subsegments until this happen; since the list is finite this is doable.)

Step 2: Now all $a_i \in U \cap V$, and $U \cap V$ is path connected. Hence we can find a path r_1 from $x_0 = f(a_0)$ to $f(a_1)$ in $U \cap V$.

Then we have two paths from the same start and endpoints in U or V, which are both simply connected, so they're homotopic.

Repeat this for all a_i, and we find that f is homotopic to a loop $r_1 * r_2 * \cdots * r_n$ in $U \cap V$. Hence it is homotopic to a loop strictly in U or V, both of which are simply connected, so f is homotopic to the constant map x_0. Hence $\pi_1(X, x_0) = 0$. □

Theorem 23.2.2. *For $n \geq 2$, the n-sphere S^n is simply connected. I.e., $\pi_1(S^n, x_0) = 0$.*

Proof. Let $p = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the north pole in S^n and let $q = (0, 0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the south pole in S^n.

Step 1: $S^n \setminus \{ p \}$ is homeomorphic to $\mathbb{R}^n$. This is classic: place the sphere on top of the origin on $\mathbb{R}^n$ and project from the north pole via a ray through the sphere, onto the plane. In particular, the map is $f \colon S^n \setminus \{ p \} \to \mathbb{R}^n$ by

$$f(x) = f(x_1, \ldots, x_{n+1}) \coloneqq \frac{1}{1 - x_{n+1}} (x_1, x_2, \ldots, x_n)$$

for $x = (x_1, \ldots, x_{n+1})$. Since $x_{n+1} \neq 1$ (we removed the north pole), f is continuous. The inverse map of f is $g \colon \mathbb{R}^n \to S^n \setminus \{ p \}$ given by

$$g(y) = g(y_1, \ldots, y_n) = (t(y)y_1, t(y)y_2, \ldots, t(y)y_n, 1 - t(y))$$

where

$$t(y) = \frac{2}{1 + \|y\|^2}.$$

Remark 23.2.3. Similarly, $S^n \setminus \{ q \}$ is homeomorphic to $S^n \setminus \{ p \}$ by the reflection map $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})$. Hence $S^n \setminus \{ q \}$ is also homeomorphic $\mathbb{R}^n$.

Step 2: Let $U = S^n \setminus \{ p \}$ and $V = S^n \setminus \{ q \}$. Then

$$S^n = U \cup V$$

and

$$U \cap V = S^n \setminus \{ p, q \}.$$

Note that $\pi_1(U) = \pi_1(V) = \pi_1(R^n) = 0$ (since the latter is convex). Also, $U \cap V$ is homeomorphic to $\mathbb{R}^n \setminus \{ 0 \}$ which is path connected for $n \geq 2$.

By Theorem 23.2.1, $\pi_1(S^n) = 0$ for $n \geq 2$. □

Corollary 23.2.4. *For $n \geq 3$, $\mathbb{R}^n \setminus \{ 0 \}$ is simply connected.*

Remark 23.2.5. Since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$, S^1 is not homeomorphic to S^n for $n \geq 2$.

Question: the fundamental group π_1 distinguishes S^1 and S^n, $n \geq 2$. But how to distinguish say S^2 and S^3?

23.3 nth homotopy group $\pi_n(X)$

Recall $\pi_1(X)$ is the set of homotopy classes of loops $f\colon [0,1] \to X$ with $f(0) = f(1)$. In other words, we can view $f\colon S^1 \to X$.

Definition 23.3.1. We let $\pi_n(X)$ be the set of homotopy classes of $f\colon S^n \to X$. We call $\pi_n(X)$ the ***n*th homotopy group** of X.

Theorem 23.3.2. *We have*

$$\pi_n(S^n) = \begin{cases} \mathbb{Z}, & \text{if } n = m \\ 0 & \text{if } n < m. \end{cases}$$

We won't explore higher homotopic theory here, so we omit the proof. But it does tell us all S^n are not homeomorphic.

It does raise a question though: what is $\pi_n(S^m)$ for $n > m$? In general, we don't know, there are only some partial results (mostly by J. P. Serre, who in their PhD thesis studied this problem and won the Fields medal).

23.4 Fundamental group of the torus

We'd like to work out the fundamental group of the torus $T = S^1 \times S^1$. We get immediately that it is $\mathbb{Z} \times \mathbb{Z}$ from the following:

Theorem 23.4.1. *Let X and Y be topological spaces and let $x_0 \in X$ and $y_0 \in Y$. Then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the natural projections. They induce homomorphisms $p_*\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $q_*\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$.

Define $\Phi\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$\Phi([\gamma]) = (p_*([\gamma]), q_*([\gamma])).$$

Then Φ is a group homomorphism (since componentwise it is a group homomorphism).

It remains to show Φ is one-to-one and onto.

First, onto: Let $[\alpha] \in \pi_1(X, x_0)$ and $[\beta] \in \pi_1(Y, y_0)$. Define $\gamma\colon [0,1] \to X \times Y$ by $\gamma(y) = (\alpha(t), \beta(t))$. Then γ is a loop based at (x_0, y_0). Moreover

$$\Phi([\gamma]) = (p_*([\gamma]), q_*([\gamma])) = ([p \circ \gamma], [q \circ \gamma]) = ([\alpha], [\beta]),$$

so Φ is onto.

Second, one-to-one. Suppose $[\gamma] \in \ker \Phi$. Then $\Phi([\gamma]) = ([p \circ \gamma], [q \circ \gamma]) = 0$, so $p \circ \gamma \simeq x_0$ and $q \circ \gamma \simeq y_0$. Let G and H be the corresponding path homotopies.

Define $F\colon [0,1] \times [0,1] \to X \times Y$ by

$$F(s,t) = (G(s,t), H(s,t)).$$

Then F is a path homotopy between γ and (x_0, y_0). Hence $\ker \Phi = \{ 0 \}$ and Φ is one-to-one. $\square$

Corollary 23.4.2. *Let $T = S^1 \times S^1$, the torus. Then $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$.*

Lecture 24 Retraction

24.1 Fundamental group of projective space

Definition 24.1.1 (Real projective plane)**.** The ***real projective plane*** $\mathbb{R}P^2 = S^2/{\sim}$ where $x \sim -x$ in S^2. That is, antipodal points on the sphere are identified.

This is sort of hard to imagine, because the real projective plane $\mathbb{R}P^2$ can not be embedded into $\mathbb{R}^3$.

Remark 24.1.2. Since $p\colon S^2 \to S^2/{\sim}$ and S^2 is compact, $\mathbb{R}P^2$ is compact.

The quotient map p is a covering map: for $[x] \in \mathbb{R}P^2$, take a small neighbourhood U, small enough that the pullback around x and $-x$ don't meet.

Proposition 24.1.3. $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Since $p\colon S^2 \to \mathbb{R}P^2$ is a covering map, and $\pi_1(S^2) = 0$, this is a universal covering. Hence $\Phi\colon \pi_1(\mathbb{R}P^2, x_0) \to \pi^{-1}(x_0)$ is one-to-one and onto.

Note $p^{-1}(x_0) = \{\, x_0, -x_0 \,\}$. Thus $\pi_1(\mathbb{R}P^2)$ is a group of order 2. But there's only one group of order 2, so $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. □

Remark 24.1.4. Taking $\mathbb{R}P^n = S^n/{\sim}$ where $x \sim -x$ in S^n is the ***real projective n-space***. Again $p\colon S^n \to \mathbb{R}P^n$ is a covering map, and $\pi_1(S^n) = 0$ for $n \geq 2$, so it is a universal cover, so by the same argument $\pi_1(\mathbb{R}P^n, x_0) \cong \mathbb{Z}/2\mathbb{Z}$.

For $n = 1$ this argument doesn't work, since $\pi_1(S^1) = \mathbb{Z} \neq 0$. In this case $p\colon S^1 \to \mathbb{R}P^1$ is a homeomorphism, so $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$.

Exercise 24.1. For $n \geq 1$, define the *real projective space* of dimension n by $\mathbb{R}P^n = S^n/{\sim}$, where the equivalence relation is defined by $x \sim y$ if and only if $x = y$ or $x = -y$.

Remark 24.1.5. In other words, $\mathbb{R}P^n$ is obtained from S^n by identifying pairs of antipodal points. It can be regarded as the set of lines in $\mathbb{R}^{n+1}$ which pass through the origin.

(*a*) $\mathbb{R}P^n$ is a compact Hausdorff space.

(*b*) The projection $\pi\colon S^n \to \mathbb{R}P^n$ is a local homeomorphism, that is, each $x \in S^n$ has an open neighbourhood that is mapped homeomorphically by π onto an open neighbourhood of $\pi(x)$.

(*c*) $\mathbb{R}P^1$ is homeomorphic to S^1. (In fact, $\mathbb{R}P^n \cong B^n/S^{n-1} \cong S^n$ for $n = 1$.) ■

24.2 Brouwer fixed-point theorem

Definition 24.2.1 (Retract)**.** Let $A \subset X$. A ***retraction*** of X onto A is a continuous map $r\colon X \to A$ such that $r\big|_A = \mathrm{Id}_A$. If such a map r exists, we call A a ***retract*** of X.

Lemma 24.2.2. *Suppose A is a retract of X. Let $j\colon A \hookrightarrow X$ be the inclusion. Then $j_*\colon \pi_1(A, a) \to \pi_1(X, a)$ is one-to-one.*

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Proof. Suppose $r\colon X \to A$ is a retraction. Then $r \circ j = \mathrm{Id}_A$. Hence $(r \circ j)_* = \mathrm{Id}_{\pi_1(A,a)} = r_* \circ j_*$. Since the identity map is one-to-one (and onto), j_* is one-to-one. $\square$

In fact,

Exercise 24.2. Let r be a retraction of X onto A and let $x_0 \in A$. Let $i\colon A \to X$ be the inclusion map. Prove the following:

(a) $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is one-to-one.

(b) $r_*\colon \pi_1(X, x_0) \to \pi_1(A, x_0)$ is onto. ■

Example 24.2.3. The map $r\colon \mathbb{R}^2 \setminus \{ 0 \} \to S^1$ given by $r(x) = \frac{x}{\|x\|}$ is a retraction. ▲

Theorem 24.2.4. *There is no retraction of the closed unit ball $B^2 = \{ x \in \mathbb{R}^2 \mid \|x\| \le 1 \}$ onto S^1.*

Proof. Suppose there is a retraction $r\colon B^2 \to S^1$. Let $j\colon S^1 \hookrightarrow B^2$ be then inclusion. Then $r \circ j\colon S^1 \hookrightarrow B^2 \to S^1$ is the identity map. Hence $r_* \circ j_* = \mathrm{Id}\colon \pi_1(S^1) \hookrightarrow \pi_1(B^2) \to \pi_1(S^1)$. But $\pi_1(S^2) = \mathbb{Z}$ and $\pi_1(B^2) = 0$, so this is a contradiction. $\square$

Lemma 24.2.5. *Let $h\colon S^1 \to X$ be a continuous map. Then the following are equivalent:*

(i) *h is* **nullhomotopic** *(i.e., $h \simeq x_0$, a constant map).*

(ii) *h extends to a continuous map $k\colon B^2 \to X$.*

(iii) *$h_*\colon \pi_1(S^1) \to \pi_1(X)$ is the trivial map (i.e., $h_* = 0$ sends everything to the identity).*

Proof. First, (i) implies (ii). Assume $h \simeq x_0$, and let $H\colon S^1 \times [0,1] \to X$ be a homotopy between h and x_0. In other words, $H(s,0) = h(s)$ for all $s \in S^1$ and $H(s,1) = x_0$ for all $s \in S^1$.

Now imagine projecting $S^1 \times [0,1]$ to B^2. So define $\pi\colon S^1 \times [0,1] \to B^2$ by $\pi(s,t) = (1-t)s$. (So identify the top circle of the cylindrical shell $S^1 \times [0,1]$ as the centre of B^2).

Notice that π is continuous and onto, and that $\pi(s,0) = s$ for all $s \in S^1$, and $\pi(s,1) = 0$.

Now define $k\colon B^2 \to X$ by $k(b) = H(b')$ for any $b' \in \pi^{-1}(b)$. Aside from the origin, $\pi^{-1}(b)$ is only one point, so no trouble there, and for the origin $b = 0$, the pullback is a circle, but that circle is homotopic to x_0 so it remains well-defined.

Since $\pi(s,0) = s$, k extends h. (Try drawing the graph of this situation—it makes it clear.)

Next (ii) implies (iii): Assume h extends to $k\colon B^2 \to X$. Let $j\colon S^1 \hookrightarrow B^2$ be the inclusion. Then

$$\begin{tikzcd} S^1 \arrow[r, hook, "j"] \arrow[rr, bend right, "h"'] & B^2 \arrow[r, "k"] & X. \end{tikzcd}$$

Then

$$h_* = k_* \circ j_*\colon \pi_1(S^1) \to \pi_1(B^2) \to \pi_1(X),$$

but $\pi_1(B^2) = 0$ and k_* is a homomorphism, so 0 must map to 0. Thus $h_* = 0$.

Finally, (iii) implies (i). Assume $h_* = 0$. Note $h_* \colon \pi_1(S^1) = \mathbb{Z} \to \pi_1(X)$. Let $p_0 \colon [0,1] \to S^1$, $p_0(r) = e^{2\pi i r}$. Then $[p_0]$ generates $\pi_1(S^1)$ (it is the single anticlockwise loop, corresponding to $1 \in \mathbb{Z}$).

Fix a point $s_0 \in S^1$ and let $x_0 = h(s_0)$.

Since $h_* = 0$, $f := h \circ p_0$ represents the identity in $\pi_1(X, x_0)$, since $h_*(p_0) = [h \circ p_0]$ by definition.

Hence the image of f is homotopic to x_0, so h is homotopic to x_0. In other words there is a path homotopy Fin X between f and x_0, i.e., $F \colon [0,1] \times [0,1] \to X$ with $F(s,0) = x_0$ and $F(s,1) = f(s)$.

Note that since F is a path homotopy, $F(0,t) = F(1,t) = x_0$ for all t. Hence F decomposes as

$$F \colon [0,1] \times [0,1] \xrightarrow{p_0 \times I} S^2 \times [0,1] \xrightarrow{G} X$$

where G is a homotopy between x_0 and h. □

Corollary 24.2.6. *(i) The inclusion $j \colon S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ is* not *nullhomotopic.*

(ii) The identity map $i \colon S^1 \to S^1$ is not *nullhomotopic.*

Proof. (i) Define the retraction $r \colon \mathbb{R}^2 \setminus \{0\} \to S^1$ by $r(x) = \frac{x}{\|x\|}$. Then

$$\begin{tikzcd} S^1 \arrow[r, hook, "j"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathbb{R}^2 \setminus \{0\} \arrow[r, "r"] & S^1. \end{tikzcd}$$

So $r_* \circ j_* = \mathrm{Id}_{\pi_1(S^1)}$. Thus j_* is one-to-one. If j is nullhomotopic, then by Lemma 24.2.5(i), $j_* = 0$, which is a contradiction. Hence j is *not* nullhomotoic.

(ii) More or less the same argument, $i \colon S^1 \to S^1$ means $i_* = \mathrm{Id}_{\pi_1(S^1)}$ so i is not nullhomotopic. □

Theorem 24.2.7. *For any nonvanishing continuous vector field on B^2, there exists a point on S^1 where the vector field points directly inward and a point on S^1 where the vector field points directly outward.*

Remark 24.2.8. A ***continuous vector field*** on B^2 is a pair $(x, v(x))$ where $x \in B^2$ and $v(x) \in \mathbb{R}^2$, with $v(x)$ continuous.

That the vector field is ***nonvanishing*** means $v(x) \neq 0$ for all x.

This theorem is equivalent to the Brouwer fixed-point theorem.

Lecture 25 Brouwer fixed-point theorem

25.1 Brouwer fixed-point theorem

Proof of Theorem 24.2.7. Suppose $v(x)$ does not point directly inward for any $x \in S^1$. Since the vector field is nonvanishing, $v \colon B^2 \to \mathbb{R}^2 \setminus \{0\}$. Let $w = v\big|_{S^1}$.

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Then by construction v is an extension of w to B^2, so by Lemma 24.2.5, w is nullhomotopic.

On the other hand, let $j\colon S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ be the inclusion map. We claim w is homotopic to j. Then $w \simeq 0$ and $w \simeq j$, so $0 \simeq j$, but this is a contradiction since $j \not\simeq 0$.

Let $F\colon S^1 \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$,

$$F(x,t) = tx + (1-t)w(x),$$

so the sort of convex, straight-line homotopy between $w(x)$ and $j(x) = x$ (thinking of $j(x)$ as the outward unit normal of S^1) since if $t = 0$, $F(x,0) = w(x)$. If $t = 1$, $F(x,1) = x = j(x)$. To make sure this is actually a homotopy, we must make sure $F(x,t) \neq 0$ for all x and t.

Suppose $F(x,t) = 0$ for some $0 < t < 1$. Then $tx + (1-t)w(x) = 0$, or

$$w(x) = -\frac{t}{1-t}x.$$

But x is radius of the unit circle, so this $w(x)$ points directly inward at x, which is a contradiction.

For $w(x)$ pointing directly outward at some point $x \in S^1$, we can consider the vector field $(x, -v(x))$, where the $-$ switches directly inward for directly outward. ☐

Theorem 25.1.1 (Brouwer fixed-point theorem). *Let $f\colon B^2 \to B^2$ be a continuous function. Then there exists a point $x \in B^2$ such that $f(x) = x$ (i.e., a* ***fixed-point****).*

Proof. Suppose $f(x) \neq x$ for all $x \in B^2$. Let $v(x) \coloneqq f(x) - x$. Then $(x, v(x))$ is a nonvanishing continuous vector field on B^2.

By Theorem 24.2.7 there exists an $x \in S^1$ such that $v(x) = f(x) - x = kx$ points directly outward, so $k > 0$.

Hence $f(x) = (k+1)x \in B^2$, and $f\colon B^2 \to B^2$, but

$$\|f(x)\| = (k+1)\|x\| = k+1 > 1$$

which is a contradiction. ☐

Remark 25.1.2. In general, every continuous map $f\colon B^{n+1} \to B^{n+1}$, $n \geq 1$, has a fixed point, i.e., there exists $x \in B^{n+1}$ such that $f(x) = x$. This is the general form of the Brouwer fixed-point theorem. (It's also true for $B^1 \to B^1$, but here it's much easier: it's just the intermediate value theorem.)

This is a consequence of there in general being no retraction $r\colon B^{n+1} \to S^n$ for $n \geq 2$. But it takes more work to show this, requiring higher homotopy, or homology.

But once we have such a result, the proof of the $n = 1$ case can be generalised.

25.2 Application of the Brouwer fixed-point theorem

Corollary 25.2.1. *Let $A \in M_3(\mathbb{R})$ with positive entries. Then A has a positive real eigenvalue.*

Proof. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the corresponding linear map to A, $T(v) = Av$. Let $B = S^2 \cap \{\, (x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0 \,\}$. (So the part of the unit sphere in the first orthant.)

Exercise 25.1. B is homeomorphic to B^2. ■

Hence the Brouwer fixed-point theorem holds for any continuous map $f\colon B \to B$.

Consider in particular $f\colon B \to B$ defined by $f(x) = \frac{T(x)}{\|T(x)\|} \in B$ (since both x and A have all positive entries).

Hence there exists some $x_0 \in B$ such that $f(x_0) = x_0$. In other words,

$$T(x_0) = \|T(x_0)\| x_0,$$

so $\|T(x_0)\| > 0$ is an eigenvalue of T, so of A, since $x_0 \neq 0$. □

25.3 The Borsuk–Ulam theorem

Definition 25.3.1. The ***antipode*** of $x \in S^n$ is the point $-x \in S^n$.

A map $h\colon S^n \to S^n$ is ***antipode-preserving*** if $h(-x) = -h(x)$ for all $x \in S^n$.

Theorem 25.3.2. *Suppose $h\colon S^1 \to S^1$ is a continuous and antipode-preserving map. Then h is not nullhomotopic.*

Proof. Let $S_0 = (1, 0) \in S^1$. Let $\rho\colon S^1 \to S^1$ be a rotation such that $\rho(h(s_0)) = s_0$.

Note that ρ is antipode-preserving. Then

$$\rho \circ h(-x) = \rho(h(-x)) = \rho(-h(x)) = -\rho(h(x)).$$

Hence the composition $\rho \circ h$ is also antipode-preserving.

If H is a homotopy between h and a constant map (so if h is nullhomotopic), then $\rho \circ H$ is a homotopy between $\rho \circ h$ and a constant map.

So we may assume $h(s_0) = s_0$ (else rotate via ρ as described).

Step 1: Define $q\colon S^1 \to S^1$ by $q(z) = z^2$, thinking of $z \in \mathbb{C}$, i.e., double the angle of z thought of in polar representation on S^1.

Note that q is a quotient map, i.e., q is continuous, onto, and open. Note also that $q^{-1}(a) = \{\, z, -z \,\}$, where $q(z) = z^2 = a$.

Second, $q(h(-z)) = q(-h(z)) = q(h(z))$. Consider the diagram

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "q"'] & S^1 \arrow[d, "q"] \\ S^1 \arrow[r, dashed, "k"'] & S^1. \end{tikzcd}$$

where we define $k\colon S^1 \to S^1$ such that $k(a) = g(h(z))$ for $z^2 = a$; since h is antipode-preserving, this is well-defined. Then

$$k \circ q = q \circ h,$$

so the diagram commute.

Note $s_0 = 1 \in \mathbb{Z}$, then $q(s_0) = s_0$ and $h(-s_0) = -h(s_0) = -s_0)$. Hence $k(s_0) = s_0$.

Step 2: we claim the induced map $h_*\colon \pi_1(S^1, s_0) \to \pi_1(S^1, s_0)$ is nontrivial.

Note that q is a covering map. Let $\tilde{f}$ be any path in S^1 from s_0 to $-s_1$. Then $f \coloneqq q \circ \tilde{f}$ is a loop based at s_0. Compute

$$k_*([f]) = [k \circ f] = [k \circ q \circ \tilde{f}] = [q \circ h \circ \tilde{f}] \neq 0$$

since $(h \circ \tilde{f})$ is a path from s_0 to $-s_0$ too, so mapped through q it's a loop covering the whole circle. Thus k_* is nontrivial.

Step 3: We claim h_* is nontrivial. Hence by Lemma 24.2.5, $h \not\simeq 0$.

We have $k_*\colon \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^1) = \mathbb{Z}$. The kernel $\ker(k_*)$ is a subgroup of $(\mathbb{Z}, +)$. If the kernel is nonzero, $\ker(k_*) = m\mathbb{Z}$ since $\mathbb{Z}$ is cyclic. Thus $\mathbb{Z}/m\mathbb{Z} \cong \operatorname{Im}(k_*)$. But $\operatorname{Im}(k_*)$ is a subgroup of $\mathbb{Z}$, and $\mathbb{Z}/m\mathbb{Z}$ is finite, but the only finite subgroup of $\mathbb{Z}$ is $\{\, 0 \,\}$, contradicting $\ker(k_*) \neq 0$.

Hence $\ker(k_*) = 0$, so k_* is one-to-one.

Note q_* is also one-to-one; $q_*\colon \pi_1(S^1) \to \pi_1(S^1)$ is the map $m \mapsto 2m$.

Thus

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "h_*"] \arrow[d, "q_*"'] & \pi_1(S^1) \arrow[d, "q_*"] \\
\pi_1(S^1) \arrow[r, "h_*"] & \pi_1(S^1)
\end{tikzcd}$$

and since every step is one-to-one, h_* cannot be trivial. □

Lecture 26 The Borsuk–Ulam theorem

26.1 The Borsuk–Ulam theorem

Theorem 26.1.1. *There is no continuous antipode-preserving map $g\colon S^2 \to S^1$.*

Proof. Suppose there exists such a $g\colon S^2 \to S^1$ which is continuous and antipode-preserving.

Take S^1 to be the equator of S^2. Then $h \coloneqq g\big|_{S^1}\colon S^1 \to S^1$ is continuous and antipode-preserving. By Theorem 25.3.2, h is not nullhomotopic. But h has an extension g the upper hemisphere E of S^2.

Moreover, $E \simeq B^2$, so by Lemma 24.2.5, h is nullhomotopic, which is a contradiction. □

Theorem 26.1.2 (Borsuk–Ulam theorem)**.** *For any continuous map $f\colon S^2 \to \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose $f(x) \neq f(-x)$ for all $x \in S^2$. Define $g\colon S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|},$$

which is defined since the denominator is never zero. Then g is continuous, and $g(-x) = -g(x)$ for all $x \in S^2$.

Hence $g\colon S^2 \to S^1$ is continuous and antipode-preserving, contradiction Theorem 26.1.1. □

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Remark 26.1.3. In general, for any continuous map $f\colon S^{n+1} \to \mathbb{R}^{n+1}$, $n \geq 0$, there exists $x \in S^{n+1}$ such that $f(x) = f(-x)$. That is, Borsuk–Ulam theorem is true in higher dimension.

The proof in general is similar, but it is much more difficult to show that if $h\colon S^{n+1} \to S^{n+1}$ is continuous and antipode-preserving, then h is not null-homotopic (that is, generalising Theorem 25.3.2). This requires either higher homotopy theory or homology.

Given this, the proof of Borsuk–Ulam theorem for $n = 1$ generalises to $n \geq 2$.

26.2 Applications

A question: Given a bounded region $A \subset \mathbb{R}^2$, can you find a line L that bisects A?

The answer is yes: consider horizontal slices of the set at $y = c$, and let $f(c)$ denote the area of the part of A below $y = c$.

Then $f(c)$ is a continuous function that varies from 0 to the area of A, so by the intermediate value theorem, it must attain half the area of A for some c.

Now a trickier question: Given two bounded regions A and B in $\mathbb{R}^2$, can you find a line L that bisects both A and B simultaneously?

The answer is still yes, but it is quite a bit less trivial.

Theorem 26.2.1 (Ham-sandwich theorem)**.** *Given any two bounded regions $A, B \subset \mathbb{R}^2$, there exists a line L in $\mathbb{R}^2$ that bisects both A and B.*

Proof. Take any two bounded regions A and B in the plane $\mathbb{R}^2 \times \{ 1 \}$, and treat this as though it is in $\mathbb{R}^3$.

Given $u \in S^2$ (i.e., corresponding to a unit vector in $\mathbb{R}^3$), let $p = p(u)$ be the plane in $\mathbb{R}^3$ passing through the origin with unit normal vector u.

Then the plane $\mathbb{R}^2 \times \{ 1 \}$ and $p(u)$ intersect in a line (unless u is vertical). Hence let $L = p \cap (\mathbb{R}^2 \times \{ 1 \})$ be this line of intersection.

Let $f_1(u)$ denote the area of the part of A that lies on side of L (say, on the same side as u). Similarly, let $f_2(u)$ be the area of the part of B that lies on the same side as u.

Note that if we replace u by $-u$, we get the same plane, only with the opposite normal. Hence $f_i(-u)$ is the area of the other side of A or B.

Hence $f_1(u) + f_1(-u)$ is the area of A, and $f_2(u) + f_2(-u)$ is the area of B.

Now consider a map $F\colon S^2 \to \mathbb{R}^2$ given by $F(u) = (f_1(u), f_2(u))$, which is continuous By Borsuk–Ulam theorem, there exists some $u \in S^2$ such that $F(u) = F(-u)$. Hence $f_1(u) = f_1(-u)$, half the area of A, and $f_2(u) = f_2(-u)$, half the area of B, meaning we have bisected A and B with the line $L = p \cap (\mathbb{R}^2 \times \{ 1 \})$. □

26.3 Invariance of domain

This is a theorem of Brouwer.

Theorem 26.3.1 (Invariance of domain)**.** *Let $U \subset \mathbb{R}^n$ be an open subset. Let $f\colon U \to \mathbb{R}^n$ be continuous and one-to-one. The $f(U)$ is open in $\mathbb{R}^n$.*

We will prove this for $n = 2$.

Lemma 26.3.2 (Homotopy extension lemma)**.** *Let X be a space such that $X \times [0,1]$ is normal. Let $A \subset X$ be a closed subset. Let $Y \subset \mathbb{R}^n$.*

Let $f\colon A \to Y$ be continuous. Suppose f is null-homotopic. Then f can be extended to a continuous map $g\colon X \to Y$ which is also nullhomotopic.

Proof. Let $F\colon A \times [0,1] \to Y$ be a homotopy between f and a constant map y_0. Hence $F(a,0) = f(a)$ for all $a \in A$, and $f(a,1) = y_0$ for all $a \in A$.

First extend F to $X \times \{1\}$ by setting $F(x,1) = y_0$ for all $x \in X$. So F is continuous on $(A \times [0,1]) \cup (X \times \{1\})$.

Note that this is a closed subset of $X \times [0,1]$, since A is closed, and this space is normal.

Use Tietze extension theorem to show that F can be extended to a continous map $G\colon X \times [0,1] \to \mathbb{R}^n$.

Exercise 26.1. Actually do this; note that our version of Tietze extension theorem only maps to $\mathbb{R}$. ■

Now $G(x,0)$ is an extension of $f(x)$, but it maps X to $\mathbb{R}^n$, not necessarily to $Y \subset \mathbb{R}^n$. We need to fix this.

Let $U = G^{-1}(Y) \subset X \times [0,1]$. Then $(A \times [0,1]) \cup (X \times \{1\}) \subset U$.

Since $[0,1]$ is compact, the ***tubular neighbourhood theorem*** implies there is an open set $W \subset X$ containing A such that $W \times [0,1] \subset U$.

Thus $G(W \times [0,1]) \subset Y$. Note $X \simeq X \times \{0\} \subset X \times [0,1]$ is a closed subset of an normal set, so X is normal.

Hence by Urysohn's lemma there exists a continuous $\phi\colon X \to [0,1]$ such that $\phi = 0$ on A and $\phi = 1$ on $X \setminus W$.

Then $x \mapsto (x, \phi(x)) \in (W \times [0,1]) \cup (X \times \{1\})$.

Thus defining $g(x) \coloneqq G(x, \phi(x))$ we have $g\colon X \to Y$.

For $x \in A$, $\phi(x) = 0$, so $g(x) = G(x,0) = f(x)$, meaning g is the desired extension of f.

Finally, we need to show $g \simeq 0$. Define $H\colon X \times [0,1] \to Y$ by $H(x,t) = G(x, (1-t)\phi(x) + t)$. Then $H(x,0) = G(x,\phi(x)) = g(x)$, and $H(x,1) = G(x,1) = F(x,1) = y_0$. □

Lecture 27 Invariance of domain

27.1 Invariance of domain

Lemma 27.1.1 (Borsuk lemma)**.** *Let $a, b \in S^2$. Let A be a compact space. Let $f\colon A \to S^2 \setminus \{a,b\}$ be a continuous injective map.*

Suppose f is nullhomotopic. Then a and b lie in the same component of $S^2 \setminus f(A)$.

Proof. Note that since A is compact and S^2 is Hausdorff, because f is one-to-one, A is homeomorphic to $f(A)$ (this is Theorem 8.1.1).

So we may assume (or think of) $f\colon A \hookrightarrow S^2 \setminus \{a,b\}$ is the inclusion map.

Note also that $S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$, so let us say $a = 0$ and $b = \infty$.

Hence we reduce to the following: Let A be a compact subset of $\mathbb{R}^2 \setminus \{0\}$. Suppose the inclusion $j\colon A \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ is nullhomotopic. Then 0 lies in the unbounded component of $\mathbb{R}^2 \setminus A$.

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To this end, let C be the connected component of $\mathbb{R}^2 \setminus A$ containing 0. We need to show that C is unbounded.

Let D be the union of the other components of $\mathbb{R}^2 \setminus A$. Then $\mathbb{R}^2 \setminus A = C \sqcup D$. Notice the left-hand side is open (since A is closed), and $\mathbb{R}^2$ is locally path connected, meaning that both C and D are open.

Suppose C is bounded. We assume $j \simeq x_0$, so by Lemma 26.3.2, j can be extended to a continuous map $k\colon C \cup A \to \mathbb{R}^2 \setminus \{ 0 \}$.

Next we extend k to $h\colon \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{ 0 \}$ by $h(x) = x$ for all $x \in D \cup A$. So $h(x) = x$ for all $x \in \mathbb{R}^2 \setminus C$.

Let $B = \{ x \in \mathbb{R}^2 \mid \|x\| \le M \}$, with M large enough that $C \cup A \subset \operatorname{int}(B)$ (which is doable since C is bounded and A is compact). Let $g = h\big|_B\colon B \to \mathbb{R}^2 \setminus \{ 0 \}$. Then $g(x) = x$ on ∂B.

Thus we have a retraction $\iota\colon B \to \partial B$, $x \mapsto M\frac{g(x)}{\|g(x)\|}$. Note in particular that $\iota(x) = x$ for $x \in \partial B$.

This is a contradiction, because there exists no retraction of B^2 to S^1 (this is Theorem 24.2.4; in fairness our argument is scaled up, but it's the same argument). $\square$

Now we are ready to prove the $n = 2$ case of Invariance of domain:

Theorem 27.1.2 (Invariance of domain)**.** *Let $U \subset \mathbb{R}^2$ be an open subset. Let $f\colon U \to \mathbb{R}^2$ be continuous and one-to-one. The $f(U)$ is open in $\mathbb{R}^2$ and $f^{-1}\colon f(U) \to U$ is continuous. (So f is an open map.)*

Proof. First we can replace $\mathbb{R}^2$ by $S^2 \cong \mathbb{R}^2 \cup \{ \infty \}$. We need to show that, for $U \subset \mathbb{R}^2$ open, if $f\colon U \to S^2$ is continuous and one-to-one, then $f(U) \subset S^2$ is open and f^{-1} is continuous.

Step 1: For any closed ball $B \subset U \subset \mathbb{R}^2$, $f(B)$ does not separate S^2. I.e., $S^2 \setminus f(B)$ only has one connected component.

Let $a, b \in S^2 \setminus f(B)$. Let $h = f\big|_B\colon B \to S^2$. Then h is nullhomotopic (consider $F(x, y) = h(tx)$, $x \in B$, $t \in [0, 1]$).

The Borsuk lemma then says a and b lie in the same component of $S^2 \setminus h(B) = S^2 \setminus f(B)$.

Step 2: For any closed ball $B \subset U \subset \mathbb{R}^2$, the image $f(\operatorname{int}(B))$ is open in S^2.

To prove this we need the following theorem:

Theorem 27.1.3 (Jordan separation theorem)**.** *Let C be a simple closed curve in S^2. Then C separates S^2. (In general, S^n separates S^{n+1}.)*

Note that $C \coloneqq f(\partial B)$ is a simple closed curve in S^2, so C separates S^2.

Let V be the connected component of $S^2 \setminus C$ that contains $f(\operatorname{int}(B))$ (which is connected since $\operatorname{int}(B)$ is connected, and f is continuous).

Let W be the union of the other components of $S^2 \setminus C$.

Since C is closed and S^2 is locally path connected, V and W are both open in S^2.

We will show $f(\operatorname{int}(B)) = V$ (and hence open).

Suppose there exists some $a \in V \setminus f(\operatorname{int}(B))$ and take any $b \in W$.

By step 1, $f(B)$ does not separate S^2, so a and b lie in the same component of $S^2 \setminus f(B) \subset S^2 \setminus C$. Hence a and b lie in the same component of $S^2 \setminus C$, which contradicts $a \in V$ and $b \in W$.

Now for any ball $B \subset U$, $f(\text{int}(B))$ is open in S^2. Hence f is an open map (on a basis, and so the whole space), and we are done. $\square$

Remark 27.1.4. There are two difficulties in generalising this for $n \geq 3$. First, there is no retraction of B^n to S^{n-1} (for which, as should now appear natural, we need higher homotopy or homology).

Second, we need the more general form of the Jordan separation theorem, saying that S^{n-1} separates S^n.

Given these two, the proof of Invariance of domain generalises nicely.

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